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On General Solutions for Field Equations in Einstein and Higher Dimension Gravity

Sergiu I. Vacaru

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Abstract We prove that the Einstein equations can be solved in a very general form for arbitrary spacetime dimensions and various types of vacuum and non-vacuum cases following a geometric method of anholonomic frame deformations for constructing exact solutions in gravity. The main idea of this method is to introduce on (pseudo) Riemannian manifolds an alternative (to the Levi-Civita connection) metric compatible linear connection which is also completely defined by the same metric structure. Such a canonically distinguished connection is with nontrivial torsion which is induced by some nonholonomy frame coefficients and generic off-diagonal terms of metrics. It is possible to define certain classes of adapted frames of reference when the Einstein equations for such an alternative connection transform into a system of partial differential equations which can be integrated in very general forms. Imposing nonholonomic constraints on generalized metrics and connections and adapted frames (selecting Levi-Civita configurations), we generate exact solutions in Einstein gravity and extra dimension generalizations.

Keywords Einstein spaces and higher dimension gravity · Anholonomic frames · Exact solutions · Nonholonomic manifolds

1 Introduction and Formulation of Main Result

The issue to construct exact solutions in Einstein gravity and high dimensional gravity theories is not new. It has been posed in different ways and related to various problems in multidimensional cosmology, black hole physics, nonlinear gravitational effects etc in Kaluza–Klein gravity and string/brane generalizations. For instance, in brane gravity, one faces the problem to generate solutions with wrapped configurations and possible quantum

S.I. Vacaru (✉)
Science Department, University "Al. I. Cuza" Iaşi, 54 Lascar Catargi street, 700107, Iaşi, Romania
e-mail: sergiu.vacaru@uaic.ro
url: http://www.scribd.com/people/view/1455460-sergiu

S.I. Vacaru
e-mail: Sergiu.Vacaru@gmail.com

corrections and noncommutative modifications. Former elaborated approaches are model dependent, usually for metric ansatz depending on 1–2 coordinates, for spherical/cylindrical symmetries and chosen backgrounds with certain types of asymptotic boundary conditions. In this context, and further application in modern physics, it is important to find a way to elaborate methods of constructed exact solutions in very general form.

The problem of constructing most general classes of solutions in gravity has been posed in a geometric language following the anholonomic deformation/frame method, see reviews of results in Refs. [1–4].[1] In brief, the method allows us to generate any spacetime metric with prescribed, or reasonable, physical/geometric properties (as solutions of Einstein equations and modifications for different gravity theories) by performing certain types of nonholonomic transforms/deformations from another well defined (pseudo) Riemannian metrics. Such a problem of constructing "almost general" solutions for Einstein spaces was solved recently for four and five dimensional spaces [5] but straightforward extensions and more sophisticate constructions should be provided to achieve such results for spacetimes of arbitrary dimensions.[2]

In this paper, we show how the Einstein equations can be integrated in general form for spacetimes of higher dimensions ${}^k n = n + m + {}^1 m + \cdots + {}^k m > 5$, when $n = 2$, or 3, and ${}^0 m = m,\ {}^1 m,\ {}^2 m, \ldots = 2$; for $k = 0, 1, 2, \ldots$. Such constructions provide a generalization of the Main Result (Theorem 1) from Ref. [5] proved for dimensions ${}^0 n = n + m = 4$, or 5 (see details on geometric methods of constructing exact solutions in gravity in Refs. [1–4]). The approach developed in this work may present a substantial interest for research in higher dimensional (super) gravity theories (which, in general, may possess higher order anisotropies [6–9]) and in higher order Lagrange–Finsler/Hamilton–Cartan geometry and related gravity and mechanical models [10–15].

Let us consider a (pseudo) Riemannian manifold ${}^k \mathbf{V}$, $\dim {}^k \mathbf{V} = {}^k n$, provided with a metric

$$\mathbf{g} = \mathbf{g}_{{}^k\alpha\ {}^k\beta}(u^{{}^k\gamma}) du^{{}^k\alpha} \otimes du^{{}^k\beta} \tag{1}$$

of arbitrary signature $\epsilon_{{}^k\alpha} = (\epsilon(1) = \pm 1, \epsilon(2) = \pm 1, \ldots, \epsilon({}^k n) = \pm 1)$.[3] The local coordinates on ${}^k \mathbf{V}$ are parametrized "shell by shell" by increasing dimensions on 2 at every level

[1] The geometry of nonholonomic distributions/deformations and frames should be not identified with the Cartan's moving frame method even in the first case "moving frames" can be also included. In our approach, we consider arbitrary real/complex, in general, noncommutative/supersymmetric nonholonomic distributions on certain manifolds and adapt the geometric constructions with respect to such distributions. Such constructions result in (nonlinear) deformations of connection and metric structures, which is not the case for moving frames, when the same geometric objects are re-expressed with respect to moving/different systems of reference. Selecting some convenient nonholonomic distributions, we obtain separations of equations and reparametrizations of variables in some physically important nonlinear systems of partial differential equations which allows us to integrate such systems in general forms. Then constraining correspondingly some general solutions, we select necessary subclasses of exact solutions, for instance, in general relativity and extra dimensions.

[2] Some conditions of theorems and formulas, for four and five dimensions, presented in Ref. [5] (that version should be considered as a Letter version of this work, where we emphasized certain techniques for generating solutions in general relativity) will be repeated for some our further constructions because they are used for different type generalizations and simplify proofs for higher dimensions.

[3] *Notation Remarks*: In our works, we follow conventions from [1, 4, 6, 7, 9] when left up/low indices are used as labels for certain types of geometric spaces/manifolds and objects. The Einstein summation rule is applied on repeating right left-up indices if it is not stated a contrary condition. Boldfaced letters are used for spaces (geometric objects) enabled with (adapted to) some nonholonomic distributions/frames prescribed on corresponding classes of manifolds. An abstract/coordinate index formalism is necessary for deriving in explicit form some general/exact solutions for gravitational field equations in higher dimensional gravity.

(equivalently, shell). We begin with denotations for ${}^{0}\mathbf{V}$, $\dim{}^{0}\mathbf{V} = {}^{0}n$, when $u^{\alpha} = (x^{i}, y^{a})$, for $u^{\alpha} = u^{{}^{0}\alpha}$ and $y^{a} = y^{{}^{0}a}$ with $k = 0$, where $x^{i} = (x^{1}, x^{\hat{i}})$ and $y^{a} = (v, y)$, i.e. $y^{4} = v$, $y^{5} = y$. Indices $i, j, k, \ldots = 1, 2, 3$; $\hat{\imath}, \hat{\jmath}, \hat{k}, \ldots = 2, 3$ and $a, b, c, \ldots = 4, 5$ are used for a conventional $(3+2)$-splitting of dimension and general abstract/coordinate indices when $\alpha, \beta, \ldots$ run values $1, 2, \ldots, 5$. For four dimensional (in brief, 4–d) constructions, we can write $u^{\hat{\alpha}} = (x^{\hat{i}}, y^{a})$, when the coordinate x^{1} and values for indices like $\alpha, i, \ldots = 1$ are not considered. In brief, we shall denote some partial derivatives $\partial_{\alpha} = \partial/\partial u^{\alpha}$ in the form $s^{\bullet} = \partial s/\partial x^{2}$, $s' = \partial s/\partial x^{3}$, $s^{*} = \partial s/\partial y^{4}$. At the next level, ${}^{1}\mathbf{V}$, $\dim{}^{1}\mathbf{V} = {}^{1}n = n + m + {}^{1}m$, the coordinates are labeled $u^{{}^{1}\alpha} = (x^{i}, y^{a}, y^{{}^{1}a})$, for ${}^{1}a,\ {}^{1}b, \ldots = 6, 7$. For the "2-anisotropy", ${}^{2}\mathbf{V}$, $\dim{}^{2}\mathbf{V} = {}^{2}n = n + m + {}^{1}m + {}^{2}m$, the coordinates are labeled $u^{{}^{2}\alpha} = (x^{i}, y^{a}, y^{{}^{1}a}, y^{{}^{2}a})$, for ${}^{2}a,\ {}^{2}b, \ldots = 8, 9$; and (recurrently) for the "k-anisotropy", ${}^{k}\mathbf{V}$, $\dim{}^{k}\mathbf{V} = {}^{k}n$, the coordinates are labeled $u^{{}^{k}\alpha} = (x^{i}, y^{a}, y^{{}^{1}a}, y^{{}^{2}a}, \ldots, y^{{}^{k}a})$, for ${}^{k}a,\ {}^{k}b, \ldots = 4 + 2k, 5 + 2k$.[4]

We shall write ${}^{k}\nabla = \{\Gamma^{{}^{k}\alpha}_{\ {}^{k}\beta\,{}^{k}\gamma}\}$ for the Levi-Civita connection, with coefficients stated with respect to an arbitrary local frame basis $e_{{}^{k}\alpha} = (e_{{}^{k-1}\alpha}, e_{{}^{k}a})$ and its dual basis $e^{{}^{k}\beta} = (e^{{}^{k-1}\beta}, e^{{}^{k}b})$. Using the Riemannian curvature tensor ${}^{k}\mathcal{R} = \{R^{{}^{k}\alpha}_{\ {}^{k}\beta\,{}^{k}\gamma\,{}^{k}\delta}\}$ defined by ${}^{k}\nabla$, one constructs the Ricci tensor, ${}^{k}\mathcal{R}ic = \{R_{{}^{k}\beta\,{}^{k}\delta} \doteqdot R^{\ {}^{k}\alpha}_{{}^{k}\beta\ {}^{k}\alpha\,{}^{k}\delta}\}$, and scalar curvature ${}^{k}R \doteqdot \mathbf{g}^{{}^{k}\beta\,{}^{k}\delta} R_{{}^{k}\beta\,{}^{k}\delta}$, where $\mathbf{g}^{{}^{k}\beta\,{}^{k}\delta}$ is inverse to $\mathbf{g}_{{}^{k}\alpha{}^{k}\beta}$. The Einstein equations on ${}^{k}\mathbf{V}$, for an energy-momentum source ${}^{k}T_{\alpha\beta}$, are written in the form

$$R_{{}^{k}\beta\,{}^{k}\delta} - \frac{1}{2}\mathbf{g}_{{}^{k}\beta\,{}^{k}\delta}\,{}^{k}R = \varkappa T_{{}^{k}\beta\,{}^{k}\delta}, \tag{2}$$

where $\varkappa = const$. For the Einstein spaces defined by a cosmological constant λ, such gravitational field equations can be represented as $R^{{}^{k}\alpha}_{\ {}^{k}\beta} = \lambda\delta^{{}^{k}\alpha}_{\ {}^{k}\beta}$, where $\delta^{{}^{k}\alpha}_{\ {}^{k}\beta}$ is the Kronecher symbol. The vacuum solutions are obtained for $\lambda = 0$.

The goal of our work is to provide (see further sections) the proof of:

Theorem 1.1 (Main Theorem) *The gravitational field equations in the ${}^{k}n$-dimensional Einstein gravity* (2) *represented by frame transforms as*

$$R^{{}^{k}\alpha}_{\ {}^{k}\beta} = \Upsilon^{{}^{k}\alpha}_{\ {}^{k}\beta}, \tag{3}$$

for any given $\Upsilon^{{}^{k}\alpha}_{\ {}^{k}\beta} = \mathrm{diag}[\Upsilon_{1}, \Upsilon_{2}, \Upsilon_{2} = \Upsilon_{3}, \Upsilon_{4}, \Upsilon_{5} = \Upsilon_{4}, \Upsilon_{6} = {}^{1}\Upsilon_{2}, \Upsilon_{7} = \Upsilon_{6}, \ldots, \Upsilon_{4+2k} = {}^{k}\Upsilon_{2}, \Upsilon_{5+2k} = \Upsilon_{4+2k}]$ *with*

$$\begin{aligned}
&\Upsilon^{1}{}_{1} = \Upsilon_{1} = \Upsilon_{2} + \Upsilon_{4}, \qquad \Upsilon^{2}{}_{2} = \Upsilon^{3}{}_{3} = \Upsilon_{2}(x^{k}, v),\\
&\Upsilon^{4}{}_{4} = \Upsilon^{5}{}_{5} = \Upsilon_{4}(x^{\hat{k}}), \qquad \Upsilon^{6}{}_{6} = \Upsilon^{7}{}_{7} = {}^{1}\Upsilon_{2}(u^{\alpha},\ {}^{1}v),\\
&\Upsilon^{8}{}_{8} = \Upsilon^{9}{}_{9} = {}^{2}\Upsilon_{2}(u^{{}^{1}\alpha},\ {}^{2}v), \quad \ldots, \quad \Upsilon^{4+2k}_{\ 4+2k} = \Upsilon^{5+2k}_{\ 5+2k} = {}^{k}\Upsilon_{2}(u^{{}^{k-1}\alpha},\ {}^{k}v),
\end{aligned} \tag{4}$$

Unfortunately, such denotations can not be introduced in a more simple form if we aim to present certain general results on exact solutions derived for some "multi-level" systems of nonlinear partial differential equations.

[4] We use the term anisotropy/anisotropic for some nonholonomically (equivalently, anholonomically) constrained variables/coordinates on a (pseudo) Riemannian manifold subjected to certain non-integrable conditions; such anisotropies should be not confused with those when geometric objects depend on some "directions and velocities", for instance, in Finsler geometry.

for $y^4 = v$, $y^6 = {}^1v$, $y^8 = {}^2v, \dots, y^{4+2k} = {}^kv$ (*where* k *labels the shell's number*), *can be solved in general form by metrics of type*

$$\begin{aligned}
{}^k\mathbf{g} &= \epsilon_1 dx^1 \otimes dx^1 + g_{\widehat{i}}(x^{\widehat{k}})dx^{\widehat{i}} \otimes dx^{\widehat{i}} + \omega^2(x^j, y^b)h_a(x^k, v)\mathbf{e}^a \otimes \mathbf{e}^a \\
&\quad + {}^1\omega^2(u^\alpha, y^{{}^1b})h_{1_a}(u^\alpha, {}^1v)\, \mathbf{e}^{{}^1a} \otimes \mathbf{e}^{{}^1a} \\
&\quad + {}^2\omega^2(u^{{}^1\alpha}, y^{{}^2b})h_{2_a}(u^{{}^1\alpha}, {}^2v)\mathbf{e}^{{}^2a} \otimes \mathbf{e}^{{}^2a} + \cdots \\
&\quad + {}^k\omega^2(u^{{}^{k-1}\alpha}, y^{{}^kb})h_{2_a}(u^{{}^{k-1}\alpha}, {}^kv)\mathbf{e}^{{}^ka} \otimes \mathbf{e}^{{}^ka},
\end{aligned} \tag{5}$$

for

$$\begin{aligned}
\mathbf{e}^4 &= dy^4 + w_i(x^k, v)dx^i, \qquad \mathbf{e}^5 = dy^5 + n_i(x^k, v)dx^i, \\
\mathbf{e}^6 &= dy^6 + w_\beta(u^\alpha, {}^1v)du^\beta, \qquad \mathbf{e}^7 = dy^7 + n_\beta(u^\alpha, {}^1v)du^\beta, \\
\mathbf{e}^8 &= dy^8 + w_{1_\beta}(u^{{}^1\alpha}, {}^2v)du^{{}^1\beta}, \qquad \mathbf{e}^9 = dy^9 + n_{1_\beta}(u^{{}^1\alpha}, {}^2v)du^{{}^1\beta}, \\
&\dots \\
\mathbf{e}^{4+2k} &= dy^{4+2k} + w_{k-1_\beta}(u^{{}^{k-1}\alpha}, {}^kv)du^{{}^{k-1}\beta}, \\
\mathbf{e}^{5+2k} &= dy^{5+2k} + n_{k-1_\beta}(u^{{}^{k-1}\alpha}, {}^kv)du^{{}^{k-1}\beta},
\end{aligned}$$

where coefficients are defined by generating functions $f(x^i, v), \partial f/\partial v \neq 0, \dots, {}^kf(u^{{}^{k-1}\alpha}, {}^kv), \partial^k f/\partial^k v \neq 0$ *and* $\omega(x^j, y^b), \dots, {}^k\omega(u^{{}^{k-1}\alpha}, y^{{}^kb}) \neq 0$ *and integration functions* ${}^0f(x^i), \dots, {}^0_kf(u^{{}^{k-1}\alpha}), {}^0h(x^i), \dots, {}^0_kh(u^{{}^{k-1}\alpha}), {}^1n_j(x^i), \dots, n_{k-1_\beta}(u^{{}^{k-1}\alpha}), {}^2n_j(x^i), \dots, {}^2_{k-1_\beta}n_j(u^{{}^{k-1}\alpha})$ *following recurrent formulas* (*when a next "shell" extends in a compatible form the previous ones*; *i.e. containing the previous constructions*), *being computed as*

$$\begin{aligned}
g_{\widehat{i}} &= \epsilon_{\widehat{i}} e^{\psi(x^{\widehat{k}})}, \quad \textit{for } \epsilon_2\psi^{\bullet\bullet} + \epsilon_3\psi'' = \Upsilon_4; \\
h_4 &= \epsilon_4\, {}^0h(x^i)[\partial_v f(x^i, v)]^2|\varsigma(x^i, v)|, \qquad h_5 = \epsilon_5[f(x^i, v) - {}^0f(x^i)]^2;
\end{aligned} \tag{6}$$

$$\begin{aligned}
w_i &= -\partial_i\varsigma(x^i, v)/\partial_v\varsigma(x^i, v), \\
n_k &= {}^1n_k(x^i) + {}^2n_k(x^i)\int dv\, \varsigma(x^i, v) \\
&\qquad [\partial_v f(x^i, v)]^2/[f(x^i, v) - {}^0f(x^i)]^3, \\
\textit{for} \quad \varsigma &= {}^0\varsigma(x^i) - \frac{\epsilon_4}{8}\, {}^0h(x^i)\int dv\, \Upsilon_2(x^k, v) \\
&\qquad \partial_v f(x^i, v)[f(x^i, v) - {}^0f(x^i)]; \\
h_6 &= \epsilon_6\, {}^0_1h(u^\alpha)[\partial_{1_v}\, {}^1f(u^\alpha, {}^1v)]^2|\, {}^1\varsigma(u^\alpha, {}^1v)|, \\
h_7 &= \epsilon_7[{}^1f(u^\alpha, {}^1v) - {}^0_1f(u^\alpha)]^2; \\
w_\beta &= -\partial_\beta\, {}^1\varsigma(u^\alpha, {}^1v)/\partial_{1_v}\, {}^1\varsigma(u^\alpha, {}^1v), \\
n_\beta &= {}^1n_\beta(u^\alpha) + {}^2n_\beta(u^\alpha)\int d\, {}^1v\, {}^1\varsigma(u^\alpha, {}^1v)
\end{aligned}$$

$$
\begin{aligned}
&[\partial_{^1v}\ {}^1f(u^\alpha,\ {}^1v)]^2/[{}^1f(u^\alpha,\ {}^1v) - {}^0_1f(u^\alpha)]^3,\\
\mathit{for}\ \ {}^1\varsigma &= {}^0_1\varsigma(u^\alpha) - \frac{\epsilon_6}{8}\,{}^0_1h(u^\alpha)\int d\,{}^1v\ {}^1\Upsilon_2(u^\alpha,\ {}^1v)\\
&[\partial_{^1v}\ {}^1f(u^\alpha,\ {}^1v)][{}^1f(u^\alpha,\ {}^1v) - {}^0_1f(u^\alpha)];\\
h_8 &= \epsilon_8\,{}^0_2h(u^{^1\alpha})\,[\partial_{^2v}\ {}^2f(u^{^1\alpha},\ {}^2v)]^2|\,{}^2\varsigma(u^{^1\alpha},\ {}^2v)|,\\
h_9 &= \epsilon_9[{}^2f(u^{^1\alpha},\ {}^2v) - {}^0_2f(u^{^1\alpha})]^2;\\
w_{^1\beta} &= -\partial_{^1\beta}\ {}^2\varsigma(u^{^1\alpha},\ {}^2v)/\partial_{^2v}\ {}^2\varsigma(u^{^1\alpha},\ {}^2v),\\
n_{^1\beta} &= {}^1n_{^1\beta}(u^{^1\alpha}) + {}^2n_{^1\beta}(u^{^1\alpha})\int d^2v\ {}^2\varsigma(u^{^1\alpha},\ {}^2v)\\
&[\partial_{^2v}\ {}^2f(u^{^1\alpha},\ {}^2v)]^2/[{}^2f(u^{^1\alpha},\ {}^2v) - {}^0_2f(u^{^1\alpha})]^3,\\
\mathit{for}\ \ {}^2\varsigma &= {}^0_2\varsigma(u^{^1\alpha}) - \frac{\epsilon_8}{8}\,{}^0_1h(u^{^1\alpha})\int d^2v\ {}^2\Upsilon_2(u^{^1\alpha},\ {}^2v)\\
&[\partial_{^2v}\ {}^2f(u^{^1\alpha},\ {}^2v)][{}^2f(u^{^1\alpha},\ {}^2v) - {}^0_2f(u^{^1\alpha})];\\
&\dots\\
h_{4+2k} &= \epsilon_{4+2k}\,{}^0_kh(u^{^{k-1}\alpha})[\partial_{^kv}\ {}^kf(u^{^{k-1}\alpha},\ {}^kv)]^2|{}^k\varsigma(u^{^{k-1}\alpha},\ {}^kv)|,\\
h_{5+2k} &= \epsilon_{5+2k}[{}^kf(u^{^{k-1}\alpha},\ {}^kv) - {}^0_kf(u^{^{k-1}\alpha})]^2;\\
w_{^{k-1}\beta} &= -\partial_{^{k-1}\beta}\ {}^k\varsigma(u^{^{k-1}\alpha},\ {}^kv)/\partial_{^kv}\ {}^k\varsigma(u^{^{k-1}\alpha},\ {}^kv),\\
n_{^{k-1}\beta} &= {}^1n_{^{k-1}\beta}(u^{^{k-1}\alpha}) + {}^2n_{^{k-1}\beta}(u^{^{k-1}\alpha})\int d\,{}^kv\ {}^k\varsigma(u^{^{k-1}\alpha},\ {}^kv)\\
&[\partial_{^kv}\ {}^kf(u^{^{k-1}\alpha},\ {}^kv)]^2/[{}^kf(u^{^{k-1}\alpha},\ {}^kv) - {}^0_kf(u^{^{k-1}\alpha})]^3,\\
\mathit{for}\ \ {}^k\varsigma &= {}^0_k\varsigma(u^{^{k-1}\alpha}) - \frac{\epsilon_{4+2k}}{8}\,{}^0_kh(u^{^{k-1}\alpha})\int d\,{}^kv\ {}^k\Upsilon_2(u^{^{k-1}\alpha},\ {}^kv)\\
&[\partial_{^kv}\ {}^kf(u^{^{k-1}\alpha},\ {}^kv)][{}^kf(u^{^{k-1}\alpha},\ {}^kv) - {}^0_kf(u^{^{k-1}\alpha})];
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{e}_k\omega &= \partial_k\omega + w_k\partial_v\omega + n_k\partial\omega/\partial y^5 = 0,\\
\mathbf{e}_\alpha\ {}^1\omega &= \partial_\alpha\ {}^1\omega + w_\alpha\partial\ {}^1\omega/\partial\ {}^1v + n_\alpha\partial\ {}^1\omega/\partial y^5 = 0,\\
\mathbf{e}_{^1\alpha}\ {}^2\omega &= \partial_{^1\alpha}\ {}^2\omega + w_{^1\alpha}\partial\ {}^2\omega/\partial\ {}^2v + n_{^1\alpha}\partial\ {}^2\omega/\partial y^7 = 0,\\
&\dots\\
\mathbf{e}_{^{k-1}\alpha}\ {}^k\omega &= \partial_{^{k-1}\alpha}\ {}^k\omega + w_{^{k-1}\alpha}\partial\ {}^k\omega/\partial\ {}^kv + n_{^{k-1}\alpha}\partial\ {}^k\omega/\partial y^{5+2k} = 0,
\end{aligned}
\tag{7}
$$

when the solutions for the Levi-Civita connection are selected by additional constraints

$$
\begin{aligned}
&\partial_v w_i = \mathbf{e}_i \ln|h_4|, \qquad \mathbf{e}_k w_i = \mathbf{e}_i w_k, \qquad \partial_v n_i = 0, \qquad \partial_i n_k = \partial_k n_i;\\
&\partial_{^1v} w_\alpha = \mathbf{e}_\alpha \ln|h_6|, \qquad \mathbf{e}_\alpha w_\beta = \mathbf{e}_\beta w_\alpha, \qquad \partial_{^1v} n_\alpha = 0, \qquad \partial_\alpha n_\beta = \partial_\beta n_\alpha;
\end{aligned}
$$

$$
\begin{aligned}
\partial_{2_v} w_{1_\alpha} &= \mathbf{e}_{1_\alpha} \ln|h_8|, \qquad \mathbf{e}_{1_\alpha} w_{1_\beta} = \mathbf{e}_{1_\beta} w_{1_\alpha}, \\
\partial_{2_v} n_{1_\alpha} &= 0, \qquad \partial_{1_\alpha} n_{1_\beta} = \partial_{1_\beta} n_{1_\alpha}; \\
&\dots \\
\partial_{k_v} w_{k-1_\alpha} &= \mathbf{e}_{1_\alpha} \ln|h_{4+2k}|, \qquad \mathbf{e}_{k-1_\alpha} w_{k-1_\beta} = \mathbf{e}_{k-1_\beta} w_{k-1_\alpha}, \\
\partial_{k_v} n_{k-1_\alpha} &= 0, \qquad \partial_{k-1_\alpha} n_{k-1_\beta} = \partial_{k-1_\beta} n_{k-1_\alpha}.
\end{aligned}
\tag{8}
$$

Following above Theorem,[5] we express the solutions of Einstein equations in high dimensional gravity in a most general form, presenting in formulas all classes of generating and integration functions and stating all constraints selecting Einstein spaces. We choose a "two by two" increasing of spacetime dimensions in formulas because this provides us a simplest way for generating "non-Killing" solutions characterized by certain types of two dimensional conformal factors depending on two "anisotropic" coordinates and the rest ones being considered as parameters. This allows us to associate to the constructed classes of solutions certain hierarchies of two-dimensional conformal symmetries with corresponding invariants and to derive associated solitonic hierarchies and bi-Hamiltonian structures as we elaborated for four dimensional spaces in Refs. [16, 17]. The length of this article does not give us a possibility to present such nonlinear wave developments for high dimensional gravity.

We have to state certain boundary/symmetry/topology conditions and define in explicit form the integration functions and systems of first order partial differential equations of type (8). This is necessary when we are interested to construct some explicit classes of exact solutions of Einstein equations (3) which are related to some physically important four and higher dimensional metrics. For instance, such high dimension solutions can be constructed to contain wormholes [18, 19] and/or to model (non) holonomic Ricci flows of various types of gravitational solitonic pp-wave, ellipsoid etc. configurations [20–24]. Perhaps all classes of exact solutions presented in the above mentioned references and (for instance, reviewed in) Refs. [1–4, 28, 29] can be found as certain particular cases of metrics (5) or certain equivalently redefined ones. In this article, we shall emphasize the geometric background of the anholonomic deformation method for construction high dimensional exact solutions in gravity and refer readers to cited works, for details and physical applications.

Any (pseudo) Riemannian metric $\mathbf{g}_{^k\alpha'\,^k\beta'}(u^{^k\gamma'})$ (1) depending in general on all $5+2k$ local coordinates on $\mathbf{V}$ can be parametrized in a form $\mathbf{g}_{^k\alpha\,^k\beta}(u^{^k\gamma})$ (5) using transforms of coefficients of metric $\mathbf{g}_{^k\alpha\,^k\beta}(u^{^k\gamma}) = e^{^k\alpha'}_{\ ^k\alpha} e^{^k\beta'}_{\ ^k\beta} \mathbf{g}_{^k\alpha'\,^k\beta'}(u^{^k\gamma'})$ under vielbein transforms $e_{^k\alpha} = e^{^k\alpha'}_{\ ^k\alpha} e_{^k\alpha'}$ preserving a chosen shell structure.[6] If the coefficients of such metrics satisfy the conditions of Main Theorem, they define general solutions of Einstein equations for any type of sources which can be parametrized in a formally diagonalized (with shell conditions) form (4), with respect to certain classes of nonholonomic frames of reference. By frame transforms such parametrizations can be defined for various types of physically important

[5] A similar Theorem was formulated in Ref. [5] for four and five dimensions; some formulas and conditions have to be repeated in this work because they are used for high dimension generalizations.

[6] We have to solve certain systems of quadratic algebraic equations and define some $e^{^k\alpha'}_{\ ^k\alpha}(u^{^k\beta})$, for given coefficients of any (5) and (1), choosing a convenient system of coordinates $u^{^k\alpha'} = u^{^k\alpha'}(u^{^k\beta})$; to present the a "shell structure" is convenient for purposes to simplify proofs of theorems; in general, under arbitrary frame/coordinate transform, all "shells" mix each with others.

energy-momentum tensors, cosmological constants (in general, with anisotropic polariziations), and for vacuum configurations.

For the case $k=0$, the proof of Theorem 1.1 is outlined in Ref. [5] using the anholonomic deformation method which was originally proposed in Refs. [25–27]. There were published a series of reviews and generalizations of the method, see [1–4], when the solutions of gravitational field equations in different types of commutative and noncommutative gravity and Ricci flow theories contain at least one Killing vector symmetry. Such higher dimension Einstein metrics with Killing symmetries, are generated if in the conditions of Theorem 1.1 there are considered $\omega,\ {}^1\omega,\ldots,\ {}^k\omega=1$.

Summarizing the results provided in Sects. 2–4 (they may be considered also as a review for the 'higher dimension' version of the anholonomic deformation method of constructing exact solutions in gravity[7]), we get a proof of Main Theorem 1.1.

Finally, we note that even we shall present a number of key results and some technical details, we shall not repeat explicit computations for coefficients of tensors and connections presented for the "Killing case" $k=0$ in our previous works [30–32], see also reviews and generalizations in [1, 2].

2 Higher Order Nonholonomic Manifolds

In this section, we outline the geometry of higher order nonholonomic manifolds which, for simplicity, will be modelled as (pseudo) Riemannian manifolds with higher order "shell" structure of dimensions ${}^k n=n+m+{}^1m+\cdots+{}^km$. Certain geometric ideas and constructions originate from the geometry of higher order Lagrange–Finsler and Hamilton–Cartan spaces defined on higher order (co) tangent classical and quantum bundles [10–13]. Such nonholonomic structures were investigated for models of (super) strings in higher order anisotropic (super) spaces [6] and for anholonomic higher order Clifford/spinor bundles [7–9]. In explicit form, the (super) gravitational gauge field equations and conservations laws were analyzed in Refs. [33–36].

2.1 Higher Order N-adapted Frames and Metrics

Our geometric spacetime arena is defined by (pseudo) Riemannian manifolds ${}^k\mathbf{V}$ enabled with nonholonomic distributions (which can be prescribed in any convenient for our purposes form like we can fix any system of reference/coordinates).

Definition 2.1 A manifold ${}^k\mathbf{V}$, $\dim {}^k\mathbf{V}={}^kn$, is higher order nonholonomic (equivalently, k-anholonomic) if its tangent bundle $T\,{}^k\mathbf{V}$ is enabled with a Whitney sum of type

$$T\,{}^k\mathbf{V}=\mathbf{h}\,{}^k\mathbf{V}\oplus v\,{}^k\mathbf{V}\oplus\ {}^1v\,{}^k\mathbf{V}\oplus\ {}^2v\,{}^k\mathbf{V}\oplus\cdots\oplus{}^kv\,{}^k\mathbf{V}. \tag{9}$$

For $k=0$, we get a usual nonholonomic manifold (or, in this case, N-anholonomic) enabled with nonlinear connection (N-connection) structure. We say that a distribution (9) defines a higher order N-connection (equivalently, ${}^k\mathbf{N}$-connection) structure.

[7] This work is organized as following: In Sect. 2, there are outlined some necessary results from the geometry of nigher order nonholonomic manifolds. Section 3 contains the system of partial differential equations (PDE) to which the Einstein equations can be transformed under nonholonomic frame transforms and deformations. In Sect. 4, we prove that it is possible to construct very general classes of exact solutions with Killing symmetries for such PDE in high dimensional gravity. We also show that it is possible to generate "non-Killing" solutions in most general forms if we consider nonholonomically deformed conformal symmetries. We conclude and discuss the results in Sect. 5.

Locally, a ${}^{k}\mathbf{N}$-connection is defined by its coefficients ${}^{k}\mathbf{N}=\{N_i^a, N_\alpha^{\ {}^1a}, N_{{}^1\alpha}^{\ {}^2a},\ldots,N_{{}^{k-1}\alpha}^{\ {}^ka}\}$, with $\{N_i^a\}\subset\{N_\alpha^{\ {}^1a}\}\subset\{N_{{}^1\alpha}^{{}^2a}\}\subset\cdots\subset N_{{}^{k-1}\alpha}^{{}^ka}\}$, when

$$ {}^0\mathbf{N}=N_i^a(u^\alpha)dx^i\otimes\frac{\partial}{\partial y^a},\qquad {}^1\mathbf{N}=N_\beta^{\ {}^1a}(u^{{}^1\alpha})du^\beta\otimes\frac{\partial}{\partial y^{{}^1a}}, $$

$$ {}^2\mathbf{N}=N_{{}^1\beta}^{{}^2a}(u^{{}^2\alpha})du^{{}^1\beta}\otimes\frac{\partial}{\partial y^{{}^2a}},\quad\ldots,\quad {}^k\mathbf{N}=N_{{}^{k-1}\beta}^{{}^ka}(u^{{}^k\alpha})du^{{}^{k-1}\beta}\otimes\frac{\partial}{\partial y^{{}^ka}}. $$

It should be noted that for general coordinate transforms on ${}^{k}\mathbf{V}$, there is a mixing of coefficients and coordinates.[8] For simplicity, we can work with adapted coordinates when some sets of coordinates on a shell of lower order are contained in a subset of coordinates on shells of higher order by trivial extensions like $u^{{}^{k-s}\alpha}\to u^{{}^{k-s+1}\alpha}=(u^{{}^{k-s}\alpha},y^{{}^{k-s+1}a})$.

Proposition 2.1 *There is a class of N-adapted frames and dual (co-) frames (equivalently, vielbeins) on* ${}^{k}\mathbf{V}$ *which depend linearly on coefficients of* ${}^{k}\mathbf{N}$*-connection.*

Proof We construct such frames following recurrent formulas for $k=0,1,\ldots,$ when

$$ \mathbf{e}_{{}^k\nu}=\left(\mathbf{e}_{{}^{k-1}\nu}=\frac{\partial}{\partial u^{{}^{k-1}\nu}}-N_{{}^{k-1}\nu}^{{}^ka}\frac{\partial}{\partial y^{{}^ka}},\,e_{{}^ka}=\partial_{{}^ka}=\frac{\partial}{\partial y^{{}^ka}}\right), \tag{10} $$

$$ \mathbf{e}^{{}^k\mu}=\left(e^{{}^{k-1}\mu}=du^{{}^{k-1}\mu},\,\mathbf{e}^{{}^ka}=dy^{{}^ka}+N_{{}^{k-1}\nu}^{{}^ka}du^{{}^{k-1}\nu}\right). \tag{11} $$

□

The vielbeins (10) satisfy the nonholonomy relations

$$ [\mathbf{e}_{{}^k\alpha},\mathbf{e}_{{}^k\beta}]=\mathbf{e}_{{}^k\alpha}\mathbf{e}_{{}^k\beta}-\mathbf{e}_{{}^k\beta}\mathbf{e}_{{}^k\alpha}=w_{{}^k\alpha\ {}^k\beta}^{{}^k\gamma}\mathbf{e}_{{}^k\gamma} \tag{12} $$

with (antisymmetric) nontrivial anholonomy coefficients $w_{{}^{k-1}\alpha\ {}^ka}^{{}^kb}=\partial N_{{}^{k-1}\alpha}^{{}^kb}/\partial u^{{}^ka}$ and $w_{{}^{k-1}\alpha\ {}^{k-1}\beta}^{{}^kb}=\Omega_{{}^{k-1}\alpha\ {}^{k-1}\beta}^{{}^kb}$, where

$$ \Omega_{{}^{k-1}\alpha\ {}^{k-1}\beta}^{{}^kb}=\mathbf{e}_{{}^{k-1}\beta}\big(N_{{}^{k-1}\alpha}^{{}^kb}\big)-\mathbf{e}_{{}^{k-1}\alpha}\big(N_{{}^{k-1}\beta}^{{}^kb}\big) \tag{13} $$

are the coefficients of curvature ${}^{k}\Omega$ of N-connection ${}^{k}\mathbf{N}$. The particular holonomic/integrable case is selected by the integrability conditions $w_{{}^{k-1}\alpha\ {}^ka}^{{}^kb}=0$.

Any (pseudo) Riemannian metric $\mathbf{g}$ on ${}^{k}\mathbf{V}$ can be written in N-adapted form, we shall write in brief that $\mathbf{g}={}^{k}\mathbf{g}=\{\mathbf{g}_{{}^k\beta\ {}^k\gamma}\}$, for

$$ \begin{aligned} {}^k\mathbf{g}&=g_{{}^{k-1}\beta\ {}^{k-1}\gamma}(u^{{}^{k-1}\alpha})e^{{}^{k-1}\beta}\otimes e^{{}^{k-1}\gamma}+h_{{}^ka\ {}^kb}(u^{{}^k\alpha})\mathbf{e}^{{}^ka}\otimes\mathbf{e}^{{}^kb}\\ &=g_{ij}(x^k)e^i\otimes e^j+h_{ab}(u^\alpha)\mathbf{e}^a\otimes\mathbf{e}^b+h_{{}^1a\ {}^1b}(u^{{}^1\alpha})\mathbf{e}^{{}^1a}\otimes\mathbf{e}^{{}^1b}\\ &\quad+h_{{}^2a\ {}^2b}(u^{{}^2\alpha})\mathbf{e}^{{}^2a}\otimes\mathbf{e}^{{}^2b}+\cdots+h_{\ {}^ka\ {}^kb}(u^{{}^k\alpha})\mathbf{e}^{{}^ka}\otimes\mathbf{e}^{{}^kb}, \end{aligned} \tag{14} $$

[8] We use boldface symbols for spaces (and geometric objects on such spaces) enabled with a structure of N-coefficients.

for some N-adapted coefficients $\mathbf{g}_{^k\beta\ ^k\gamma} = [g_{ij}, h_{ab}, h_{^1a\ ^1b}, \ldots, h_{^ka\ ^kb}]$ and $N^{^ka}_{^{k-1}\alpha}$. For constructing exact solutions in high dimensional gravity, it is convenient to work with such N-adapted formulas for tensors' and connections' coefficients.

For instance, we get from (14) a metric with a parametrization of type (5) when all $^k\omega = 1$ if we choose

$$
\begin{aligned}
g_{ij} &= \mathrm{diag}[\epsilon_1, g_{\widehat{i}}(x^{\widehat{k}})], \qquad h_{ab} = \mathrm{diag}[h_a(x^i, v)], \\
&\quad N_k^4 = w_k(x^i, v), \qquad N_k^5 = n_k(x^i, v); \\
h_{^1a\ ^1b} &= \mathrm{diag}[h_{^1a}(u^\alpha,\ ^1v)], \\
&\quad N_\beta^6 = w_\beta(u^\alpha,\ ^1v), \qquad N_\beta^7 = n_\beta(u^\alpha,\ ^1v); \\
h_{^2a\ ^2b} &= \mathrm{diag}[h_{^2a}(u^{^1\alpha},\ ^2v)], \\
&\quad N^8_{^1\beta} = w_{^1\beta}(u^{^1\alpha},\ ^2v), \qquad N^9_{^1\beta} = n_{^1\beta}(u^{^1\alpha},\ ^2v); \\
&\quad \ldots \\
h_{^ka\ ^kb} &= \mathrm{diag}[h_{^ka}(u^{^{k-1}\alpha},\ ^kv)], \\
&\quad N^{4+2k}_{^{k-1}\beta} = w_{^{k-1}\beta}(u^{^{k-1}\alpha},\ ^kv), \qquad N^{5+2k}_{^{k-1}\beta} = n_{^{k-1}\beta}(u^{^{k-1}\alpha},\ ^kv).
\end{aligned}
\tag{15}
$$

Such a metric has symmetries of $k+1$ Killing vectors, $e_5 = \partial/\partial y^5, e_7 = \partial/\partial y^7, \ldots, e_{5+2k} = \partial/\partial y^{5+2k}$, because its coefficients do not depend on $y^5, y^7, \ldots, y^{5+2k}$. Introducing nontrivial $^k\omega^2(u^{^k\alpha})$ depending also on y^{5+2k}, as multiples before $h_{^ka}$, we get N-adapted parametrizations, up to certain frame/coordinate transforms, for all metrics on $^k\mathbf{V}$. In Sect. 3, we shall define the equations which must satisfy the coefficients of N-adapted metrics when (15) will generate exact solutions of Einstein equations.

2.2 N-adapted Deformations of the Levi-Civita Connection

By straightforward computations, it is a cumbersome task to prove using the Levi-Civita connection[9] $^k\nabla$) that the Einstein equations (3) on higher dimensional spacetimes are solved by metrics of type (5). We are going to show explicitly that general solutions for $^k\nabla$ can be constructed passing three steps: (1) to adapt our constructions to N-adapted frames of type $\mathbf{e}_\alpha$ (10) and $\mathbf{e}^\mu$ (11); (2) to use as an auxiliary tool (we emphasize, in Einstein gravity and its generalizations on high dimensional (pseudo) Riemannian manifolds) a new type of linear connection $^k\widehat{\mathbf{D}} = \{\widehat{\Gamma}^{^k\alpha}_{^k\beta\ ^k\gamma}\}$, also uniquely defined by the metric structure; (3) To constrain the integral varieties of general solutions in such a form that $^k\widehat{\mathbf{D}} \to\ ^k\nabla$.

Definition 2.2 A distinguished connection $^k\mathbf{D}$ (in brief, d-connection) on $^k\mathbf{V}$ is a linear connection preserving under parallelism a conventional horizontal and k-vertical splitting (in brief, h- and v-splitting) induced by $^k\mathbf{N}$-connection structure (9).

We note that the Levi-Civita connection $^k\nabla$, for which $^k\nabla\ ^k\mathbf{g} = 0$ and $^k\mathcal{T}^{^k\alpha} \doteqdot\ ^k\nabla \mathbf{e}^{^k\alpha} = 0$, is not a d-connection because, in general, it is not adapted to a N-splitting defined by a Whitney sum (9). So, in order to elaborate self-consistent geometric/physical models adapted to a N-connection it is necessary to work with d-connections.

[9] A unique one, which is metric compatible and with zero torsion, and completely defined by the metric structure.

Theorem 2.1 *There is a unique canonical d-connection* ${}^k\widehat{\mathbf{D}}$ *satisfying the condition* ${}^k\widehat{\mathbf{D}}\,{}^k\mathbf{g} = 0$ *and with vanishing "pure" horizontal and vertical torsion coefficients, i.e.* $\widehat{T}^i{}_{jk} = 0$ *and* $\widehat{T}^{{}^k a}{}_{{}^k b\, {}^k c} = 0$, *see (below) formulas (20).*

Proof Let us define ${}^k\widehat{\mathbf{D}}$ as a 1-form

$$\widehat{\mathbf{\Gamma}}^{{}^k\alpha}{}_{{}^k\beta} = \widehat{\Gamma}^{{}^k\alpha}{}_{{}^k\beta\, {}^k\gamma}\mathbf{e}^{{}^k\gamma} \tag{16}$$

with $\widehat{\mathbf{\Gamma}}^{{}^k\gamma}{}_{{}^k\alpha\, {}^k\beta} = (\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc}; \widehat{L}^{{}^1a}_{{}^1b\alpha}, \widehat{C}^\alpha_{\beta\, {}^1c}, \widehat{C}^{{}^1a}_{{}^1b\, {}^1c}; \ldots; \widehat{L}^{{}^ka}_{{}^kb\, {}^{k-1}\alpha}, \widehat{C}^{{}^{k-1}\alpha}_{{}^{k-1}\beta\, {}^kc}, \widehat{C}^{{}^ka}_{{}^kb\, {}^kc})$, where $\widehat{L}^{{}^1\alpha}_{{}^1\beta\, {}^1\gamma} = \widehat{\Gamma}^\alpha_{\beta\gamma} = (\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc})$; $\widehat{L}^{{}^2\alpha}_{{}^2\beta\, {}^2\gamma} = \widehat{\Gamma}^{{}^1\alpha}_{{}^1\beta\, {}^1\gamma}; \ldots; \widehat{L}^{\,{}^k\alpha}_{\,{}^k\beta\, {}^k\gamma} = \widehat{\mathbf{\Gamma}}^{{}^{k-1}\alpha}_{{}^{k-1}\beta\, {}^{k-1}\gamma}$, for

$$\begin{aligned}
\mathbf{g}_{\alpha\beta} &= [g_{ij}(x^k), h_{ab}(u^\gamma)], \qquad \mathbf{g}_{{}^1\alpha\, {}^1\beta} = [\mathbf{g}_{\alpha\beta}(u^\gamma), h_{{}^1a\, {}^1b}(u^{{}^1\gamma})],\\
\mathbf{g}_{{}^2\alpha\, {}^2\beta} &= [\mathbf{g}_{{}^1\alpha\, {}^1\beta}(u^{{}^1\gamma}), h_{{}^2a\, {}^2b}(u^{{}^2\gamma})], \ldots,\\
\mathbf{g}_{{}^k\alpha\, {}^k\beta} &= [\mathbf{g}_{{}^{k-1}\alpha\, {}^{k-1}\beta}(u^{{}^{k-1}\gamma}), h_{{}^ka\, {}^kb}(u^{{}^k\gamma})],
\end{aligned}$$

where

$$\begin{aligned}
\widehat{L}^i_{jk} &= \frac{1}{2}g^{ir}(\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk}),\\
\widehat{L}^a_{bk} &= e_b(N^a_k) + \frac{1}{2}h^{ac}\big(\mathbf{e}_k h_{bc} - h_{dc}\, e_b N^d_k - h_{db}\, e_c N^d_k\big),\\
\widehat{C}^i_{jc} &= \frac{1}{2}g^{ik}e_c g_{jk}, \qquad \widehat{C}^a_{bc} = \frac{1}{2}h^{ad}(e_c h_{bd} + e_c h_{cd} - e_d h_{bc}),\\
\widehat{L}^{{}^1a}_{{}^1b\alpha} &= e_{{}^1b}(N^{{}^1a}_\alpha) + \frac{1}{2}h^{{}^1a\, {}^1c}(\mathbf{e}_\alpha h_{{}^1b\, {}^1c} - h_{{}^1d\, {}^1c}\, e_{{}^1b}N^{{}^1d}_\alpha - h_{{}^1d\, {}^1b}\, e_{{}^1c}N^{{}^1d}_\alpha),\\
\widehat{C}^\alpha_{\beta\, {}^1c} &= \frac{1}{2}g^{\alpha\gamma}e_{{}^1c}g_{\beta\gamma},\\
\widehat{C}^{{}^1a}_{{}^1b\, {}^1c} &= \frac{1}{2}h^{{}^1a\, {}^1d}(e_{{}^1c}h_{{}^1b\, {}^1d} + e_{{}^1c}h_{{}^1c\, {}^1d} - e_{{}^1d}h_{{}^1b\, {}^1c}),\\
\widehat{L}^{{}^2a}_{{}^2b\, {}^1\alpha} &= e_{{}^2b}(N^{{}^2a}_{{}^1\alpha}) + \frac{1}{2}h^{{}^2a\, {}^2c}(\mathbf{e}_{{}^1\alpha}h_{{}^2b\, {}^2c} - h_{{}^2d\, {}^2c}\, e_{{}^2b}N^{{}^2d}_{{}^1\alpha} - h_{{}^2d\, {}^2b}\, e_{{}^2c}N^{{}^2d}_{{}^1\alpha}),\\
\widehat{C}^{{}^1\alpha}_{{}^1\beta\, {}^2c} &= \frac{1}{2}g^{{}^1\alpha\, {}^1\gamma}e_{{}^2c}g_{{}^1\beta\, {}^1\gamma},\\
\widehat{C}^{{}^2a}_{{}^2b\, {}^2c} &= \frac{1}{2}h^{{}^2a\, {}^2d}(e_{{}^2c}h_{{}^2b\, {}^2d} + e_{{}^2c}h_{{}^2c\, {}^2d} - e_{{}^2d}h_{{}^2b\, {}^2c}),\\
&\ldots\\
\widehat{L}^{{}^ka}_{{}^kb\, {}^{k-1}\alpha} &= e_{{}^kb}(N^{{}^ka}_{{}^{k-1}\alpha}) + \frac{1}{2}h^{{}^ka\, {}^kc}(\mathbf{e}_{{}^{k-1}\alpha}h_{{}^kb\, {}^kc} - h_{{}^kd\, {}^kc}\, e_{{}^kb}N^{{}^kd}_{{}^{k-1}\alpha} - h_{{}^kd\, {}^kb}e_{{}^kc}N^{{}^kd}_{{}^{k-1}\alpha}),\\
\widehat{C}^{{}^{k-1}\alpha}_{{}^{k-1}\beta\, {}^kc} &= \frac{1}{2}g^{{}^{k-1}\alpha\, {}^{k-1}\gamma}e_{{}^kc}g_{{}^{k-1}\beta\, {}^{k-1}\gamma},\\
\widehat{C}^{{}^ka}_{{}^kb\, {}^kc} &= \frac{1}{2}h^{{}^ka\, {}^kd}(e_{{}^kc}h_{{}^kb\, {}^kd} + e_{{}^kc}h_{{}^kc\, {}^kd} - e_{{}^kd}h_{{}^kb\, {}^kc}).
\end{aligned} \tag{17}$$

It follows by straightforward verifications that ${}^k\widehat{\mathbf{D}}\,{}^k\mathbf{g} = 0$, where this N-adapted metric compatibility condition splits into

$$
\begin{aligned}
&\widehat{D}_j g_{kl} = 0, && \widehat{D}_a g_{kl} = 0, && \widehat{D}_j h_{ab} = 0, && \widehat{D}_a h_{bc} = 0,\\
&\widehat{D}_\gamma g_{\alpha\beta} = 0, && \widehat{D}_{{}^1a} g_{\alpha\beta} = 0, && \widehat{D}_\gamma h_{{}^1a\,{}^1b} = 0, && \widehat{D}_{{}^1a} h_{{}^1b\,{}^1c} = 0,\\
&\widehat{D}_{{}^1\gamma} g_{{}^1\alpha\,{}^1\beta} = 0, && \widehat{D}_{{}^2a} g_{{}^1\alpha\,{}^1\beta} = 0, && \widehat{D}_{{}^1\gamma} h_{{}^2a\,{}^2b} = 0, && \widehat{D}_{{}^2a} h_{{}^2b\,{}^2c} = 0,\\
&\dots\\
&\widehat{D}_{{}^{k-1}\gamma} g_{{}^{k-1}\alpha\,{}^{k-1}\beta} = 0, && \widehat{D}_{{}^ka} g_{{}^{k-1}\alpha\,{}^{k-1}\beta} = 0,\\
&\widehat{D}_{{}^{k-1}\gamma} h_{{}^ka\,{}^kb} = 0, && \widehat{D}_{{}^ka} h_{{}^kb\,{}^kc} = 0,
\end{aligned}
\tag{18}
$$

where the covariant derivatives are computed using corresponding coefficients, step by step, on every shell. The canonical d-connection contains an induced torsion (completely determined by the coefficients of metric, and respective N-connection coefficients)

$$
\widehat{\mathcal{T}}^{{}^k\alpha} = \widehat{\mathbf{T}}^{{}^k\alpha}{}_{{}^k\beta\,{}^k\gamma}\mathbf{e}^{{}^k\beta} \wedge \mathbf{e}^{{}^k\gamma} \doteqdot {}^k\widehat{\mathbf{D}}\mathbf{e}^{{}^k\alpha} = d\mathbf{e}^{{}^k\alpha} + \widehat{\Gamma}^{{}^k\alpha}{}_{{}^k\beta} \wedge \mathbf{e}^{{}^k\beta},
\tag{19}
$$

with coefficients

$$
\begin{aligned}
&\widehat{T}^i{}_{jk} = \widehat{L}^i{}_{jk} - \widehat{L}^i{}_{kj}, \qquad \widehat{T}^i{}_{ja} = -\widehat{T}^i{}_{aj} = \widehat{C}^i{}_{ja}, \qquad T^a{}_{ji} = -\Omega^a{}_{ji},\\
&\widehat{T}^a{}_{bi} = -\widehat{T}^a{}_{ib} = \frac{\partial N^a_i}{\partial y^b} - \widehat{L}^a{}_{bi}, \qquad \widehat{T}^a{}_{bc} = \widehat{C}^a{}_{bc} - \widehat{C}^a{}_{cb};\\
&\widehat{T}^\alpha{}_{\beta\gamma} = \widehat{L}^\alpha{}_{\beta\gamma} - \widehat{L}^\alpha{}_{\gamma\beta}, \qquad \widehat{T}^\alpha{}_{\beta\,{}^1a} = -\widehat{T}^\alpha{}_{{}^1a\beta} = \widehat{C}^\alpha{}_{\beta\,{}^1a}, \qquad T^{\,{}^1a}{}_{\beta\alpha} = -\Omega^{\,{}^1a}{}_{\beta\alpha},\\
&\widehat{T}^{\,{}^1a}{}_{{}^1b\alpha} = -\widehat{T}^{\,{}^1a}{}_{\alpha\,{}^1b} = \frac{\partial N_\alpha^{\,{}^1a}}{\partial y^{{}^1b}} - \widehat{L}^{\,{}^1a}{}_{{}^1b\alpha}, \qquad \widehat{T}^{{}^1a}{}_{{}^1b\,{}^1c} = \widehat{C}^{{}^1a}{}_{{}^1b\,{}^1c} - \widehat{C}^{{}^1a}{}_{{}^1c\,{}^1b};\\
&\widehat{T}^{{}^1\alpha}{}_{{}^1\beta\,{}^1\gamma} = \widehat{L}^{\,{}^1\alpha}{}_{{}^1\beta\,{}^1\gamma} - \widehat{L}^{\,{}^1\alpha}{}_{{}^1\gamma\,{}^1\beta}, \qquad \widehat{T}^{\,{}^1\alpha}{}_{{}^1\beta\,{}^2a} = -\widehat{T}^{\,{}^1\alpha}{}_{{}^2a\,{}^1\beta} = \widehat{C}^{\,{}^1\alpha}{}_{{}^1\beta\,{}^2a},\\
&T^{{}^2a}{}_{{}^1\beta\,{}^1\alpha} = -\Omega^{{}^2a}{}_{{}^1\beta\,{}^1\alpha}, \qquad \widehat{T}^{{}^2a}{}_{{}^2b\,{}^1\alpha} = -\widehat{T}^{{}^2a}{}_{{}^1\alpha\,{}^2b} = \frac{\partial N^{{}^2a}_{{}^1\alpha}}{\partial y^{{}^2b}} - \widehat{L}^{{}^2a}{}_{{}^2b\,{}^1\alpha},\\
&\widehat{T}^{{}^2a}{}_{{}^2b\,{}^2c} = \widehat{C}^{{}^2a}{}_{{}^2b\,{}^2c} - \widehat{C}^{{}^2a}{}_{{}^2c\,{}^2b};\\
&\dots\\
&\widehat{T}^{{}^{k-1}\alpha}{}_{{}^{k-1}\beta\,{}^{k-1}\gamma} = \widehat{L}^{{}^{k-1}\alpha}{}_{{}^{k-1}\beta\,{}^{k-1}\gamma} - \widehat{L}^{{}^{k-1}\alpha}{}_{{}^{k-1}\gamma\,{}^{k-1}\beta},\\
&\widehat{T}^{{}^{k-1}\alpha}{}_{{}^{k-1}\beta\,{}^ka} = -\widehat{T}^{{}^{k-1}\alpha}{}_{{}^ka\,{}^{k-1}\beta} = \widehat{C}^{{}^{k-1}\alpha}{}_{{}^{k-1}\beta\,{}^ka}, \qquad T^{{}^ka}{}_{{}^{k-1}\beta\,{}^{k-1}\alpha} = -\Omega^{{}^ka}{}_{{}^{k-1}\beta\,{}^{k-1}\alpha},\\
&\widehat{T}^{{}^ka}{}_{{}^kb\,{}^{k-1}\alpha} = -\widehat{T}^{\,{}^ka}{}_{{}^{k-1}\alpha\,{}^kb} = \frac{\partial N^{{}^ka}_{{}^{k-1}\alpha}}{\partial y^{\,{}^kb}} - \widehat{L}^{{}^ka}{}_{{}^kb\,{}^{k-1}\alpha},\\
&\widehat{T}_{{}^kb\,{}^kc}{}^{{}^ka} = \widehat{C}^{{}^ka}{}_{{}^kb\,{}^kc} - \widehat{C}^{{}^ka}{}_{{}^kc\,{}^kb}.
\end{aligned}
\tag{20}
$$

Introducing values (17) into (20) we get that $\widehat{T}^i{}_{jk} = 0$ and $\widehat{C}^{{}^ka}{}_{{}^kb\,{}^kc} = 0$ which satisfy the conditions of this theorem. In general, other N-adapted torsion coefficients (for instance, $\widehat{T}^i{}_{ja}$, $\widehat{T}^a{}_{ji}$ and $\widehat{T}^a{}_{bi}$) are not zero. □

The torsion (19) is very different from that, for instance, in Einstein–Cartan, string, or gauge gravity because we do not consider additional field equations (algebraic or dynamical ones), see discussions in [2, 3]. In our case, the nontrivial torsion coefficients are related to anholonomy coefficients $w^{\ ^k\gamma}_{\ ^k\alpha\ ^k\beta}$ in (12).

We can distinguish the covariant derivative $^k\widehat{\mathbf{D}}$ determined by formulas (16) and (17), in N-adapted to (9) form, as $\widehat{\mathbf{D}}_{^k\alpha} = (\widehat{D}_i, \widehat{D}_a, \widehat{D}_{^1a}, \ldots, \widehat{D}_{^ka})$, where $\widehat{D}_{^ka}$ are shell operators.

From Theorem 2.1, we get:

Corollary 2.1 *Any geometric construction for the canonical d-connection* $^k\widehat{\mathbf{D}} = \{\widehat{\Gamma}^{\ ^k\gamma}_{\ ^k\alpha\ ^k\beta}\}$ *can be re-defined equivalently into a similar one with the Levi-Civita connection* $^k\nabla = \{\Gamma^{\ ^k\gamma}_{\ ^k\alpha\ ^k\beta}\}$ *following formulas*

$$\Gamma^{\ ^k\gamma}_{\ ^k\alpha\ ^k\beta} = \widehat{\Gamma}^{\ ^k\gamma}_{\ ^k\alpha\ ^k\beta} + Z^{\ ^k\gamma}_{\ ^k\alpha\ ^k\beta}, \tag{21}$$

where the N-adapted coefficients of linear connections, $\Gamma^{\ ^k\gamma}_{\ ^k\alpha\ ^k\beta}$, $\widehat{\mathbf{\Gamma}}^{\ ^k\gamma}_{\ ^k\alpha\ ^k\beta}$, *and the distortion tensor* $Z^{\ ^k\gamma}_{\ ^k\alpha\ ^k\beta}$ *are determined in unique forms by the coefficients of a metric* $\mathbf{g}_{^k\alpha\ ^k\beta}$.

Proof It is similar to that presented for vector bundles in Refs. [14, 15] but in our case adapted for (pseudo) Riemannian nonholonomic manifolds, see details in [1, 3, 4] and, in higher order form, in [6, 7, 9]. Here we present the N-adapted components of the distortion tensor $Z^{\ ^k\gamma}_{\ ^k\alpha\ ^k\beta}$ computed as

$$\begin{aligned}
&Z^a_{jk} = -\widehat{C}^i_{jb} g_{ik} h^{ab} - \frac{1}{2}\Omega^a_{jk}, \qquad Z^i_{bk} = \frac{1}{2}\Omega^c_{jk} h_{cb} g^{ji} - \Xi^{ih}_{jk}\widehat{C}^j_{hb},\\
&Z^a_{bk} = {}^{+}\Xi^{ab}_{cd}\widehat{T}^c_{kb}, \qquad Z^i_{kb} = \frac{1}{2}\Omega^a_{jk} h_{cb} g^{ji} + \Xi^{ih}_{jk}\widehat{C}^j_{hb}, \qquad Z^i_{jk} = 0,\\
&Z^a_{jb} = -{}^{-}\Xi^{ad}_{cb}\widehat{T}^c_{jd}, \qquad Z^a_{bc} = 0, \qquad Z^i_{ab} = -\frac{g^{ij}}{2}\left[\widehat{T}^c_{ja} h_{cb} + \widehat{T}^c_{jb} h_{ca}\right],\\
&Z^{^1a}_{\beta\gamma} = -\widehat{C}^{\alpha}_{\beta ^1b} g_{\alpha\gamma} h^{^1a\ ^1b} - \frac{1}{2}\Omega^{\ ^1a}_{\beta\gamma}, \qquad Z^{\alpha}_{\ ^1b\gamma} = \frac{1}{2}\Omega^{\ ^1c}_{\beta\gamma} h_{^1c\ ^1b} g^{\beta\alpha} - \Xi^{\alpha\tau}_{\beta\gamma}\widehat{C}^{\beta}_{\tau\ ^1b},\\
&Z^{^1a}_{\ ^1b\gamma} = {}^{+}\Xi^{^1a\ ^1b}_{^1c\ ^1d}\widehat{T}^{\ ^1c}_{\gamma\ ^1b}, \qquad Z^{\alpha}_{\beta\ ^1b} = \frac{1}{2}\Omega^{\ ^1a}_{\beta\gamma} h_{^1c\ ^1b} g^{\beta\alpha} + \Xi^{\alpha\tau}_{\beta\gamma}\widehat{C}^{\beta}_{\tau\ ^1b},\\
&Z^{\alpha}_{\beta\gamma} = 0, \qquad Z^{\ ^1a}_{\beta\ ^1b} = -{}^{-}\Xi^{^1a\ ^1d}_{^1c\ ^1b}\widehat{T}^{\ ^1c}_{\beta\ ^1d}, \qquad Z^{^1a}_{\ ^1b\ ^1c} = 0,\\
&Z^{\alpha}_{\ ^1a\ ^1b} = -\frac{g^{\alpha\beta}}{2}\left[\widehat{T}^{\ ^1c}_{\beta\ ^1a} h_{^1c\ ^1b} + \widehat{T}^{\ ^1c}_{\beta\ ^1b} h_{^1c\ ^1a}\right],\\
&Z^{^2a}_{\ ^1\beta\ ^1\gamma} = -\widehat{C}^{^1\alpha}_{\ ^1\beta\ ^2b} g_{^1\alpha\ ^1\gamma} h^{^2a\ ^2b} - \frac{1}{2}\Omega^{\ ^2a}_{^1\beta\ ^1\gamma}, \qquad Z^{^1\alpha}_{\ ^2b\ ^1\gamma} = \widehat{L}^5_{4j} = \frac{1}{2}\partial_v n_j,\\
&\frac{1}{2}\Omega^{\ ^2c}_{^1\beta\ ^1\gamma} h_{^2c\ ^2b} g^{^1\beta\ ^1\alpha} - \Xi^{^1\alpha\ ^1\tau}_{^1\beta\ ^1\gamma}\widehat{C}^{^1\beta}_{\ ^1\tau\ ^2b}, \qquad Z^{^2a}_{\ ^2b\ ^1\gamma} = {}^{+}\Xi^{^2a\ ^2b}_{^2c\ ^2d}\widehat{T}^{^2c}_{\ ^1\gamma\ ^2b},\\
&Z^{^1\alpha}_{\ ^1\beta\ ^2b} = \frac{1}{2}\Omega^{\ ^2a}_{^1\beta\ ^1\gamma} h_{^2c\ ^2b} g^{^1\beta\ ^1\alpha} + \Xi^{^1\alpha\ ^1\tau}_{^1\beta\ ^1\gamma}\widehat{C}^{^1\beta}_{\ ^1\tau\ ^2b},\\
&Z^{^1\alpha}_{\ ^1\beta\ ^1\gamma} = 0, \qquad Z^{^2a}_{\ ^1\beta\ ^2b} = -{}^{-}\Xi^{^2a\ ^2d}_{^2c\ ^2b}\widehat{T}^{^2c}_{\ ^1\beta\ ^2d}, \qquad Z^{^2a}_{\ ^2b\ ^2c} = 0,
\end{aligned} \tag{22}$$

$$Z^{1_\alpha}_{2_a\ 2_b} = -\frac{g^{1_\alpha\ 1_\beta}}{2}\big[\widehat{T}^{2_c}_{1_\beta\ 2_a} h_{2_c\ 2_b} + \widehat{T}^{2_c}_{1_\beta\ 2_b} h_{2_c\ 2_a}\big],$$

$$\dots$$

$$Z^{k_a}{}_{k-1_\beta\ k-1_\gamma} = -\widehat{C}^{k-1_\alpha}_{k-1_\beta\ k_b} g_{k-1_\alpha\ k-1_\gamma} h^{k_a\ k_b} - \frac{1}{2}\Omega^{k_a}_{k-1_\beta\ k-1_\gamma},$$

$$Z^{k-1_\alpha}_{k_b\ k-1_\gamma} = \frac{1}{2}\Omega^{k_c}_{k-1_\beta\ k-1_\gamma} h_{k_c\ k_b} g^{k-1_\beta\ k-1_\alpha} - \Xi^{k-1_\alpha\ k-1_\tau}_{k-1_\beta\ k-1_\gamma} \widehat{C}^{k-1_\beta}_{k-1_\tau\ k_b},$$

$$Z^{k_a}_{k_b\ k-1_\gamma} = {}^{+}\Xi^{k_a\ k_b}_{k_c\ k_d} \widehat{T}^{k_c}_{k-1_\gamma\ k_b},$$

$$Z^{k-1_\alpha}_{k-1_\beta\ k_b} = \frac{1}{2}\Omega^{k_a}_{k-1_\beta\ k-1_\gamma} h_{k_c\ k_b} g^{k-1_\beta\ k-1_\alpha} + \Xi^{k-1_\alpha\ k-1_\tau}_{k-1_\beta\ k-1_\gamma} \widehat{C}^{k-1_\beta}_{k-1_\tau\ k_b},$$

$$Z^{k-1_\alpha}_{k-1_\beta\ k-1_\gamma} = 0, \qquad Z^{k_a}_{k-1_\beta\ k_b} = -{}^{-}\Xi^{k_a\ k_d}_{k_c\ k_b} \widehat{T}^{k_c}_{k-1_\beta\ k_d}, \qquad Z^{k_a}_{k_b\ k_c} = 0,$$

$$Z^{k-1_\alpha}_{k_a\ k_b} = -\frac{g^{k-1_\alpha\ k-1_\beta}}{2}\big[\widehat{T}^{k_c}_{k-1_\beta\ k_a} h_{k_c\ k_b} + \widehat{T}^{k_c}_{k-1_\beta\ k_b} h_{k_c\ k_a}\big],$$

$$\text{for}\quad \Xi^{ih}_{jk} = \frac{1}{2}(\delta^i_j \delta^h_k - g_{jk} g^{ih}), \qquad {}^{\pm}\Xi^{ab}_{cd} = \frac{1}{2}(\delta^a_c \delta^b_d \pm h_{cd} h^{ab}),$$

$$^{\pm}\Xi^{1_a\ 1_b}_{1_c\ 1_d} = \frac{1}{2}(\delta^{1_a}_{1_c}\delta^{1_b}_{1_d} \pm h_{1_c\ 1_d} h^{1_a\ 1_b}), \dots,$$

$$^{\pm}\Xi^{k_a\ k_b}_{k_c\ k_d} = \frac{1}{2}(\delta^{k_a}_{k_c}\delta^{k_b}_{k_d} \pm h_{k_c\ k_d} h^{k_a\ k_b}),$$

where the necessary torsion coefficients are computed as in (20). □

Remark 2.1 Hereafter, we shall omit certain details on shell components of formulas and computations if that will not result in ambiguities. Such constructions are similar to those presented in above Theorems and in Refs. [1–4, 6–9, 30–32, 34–36]. Some additional necessary formulas are given in Appendix.

In four dimensions, the Einstein gravity can be equivalently formulated in the so-called almost Kähler and Lagrange–Finsler variables, as we considered in Refs. [4, 37–39]. Similarly, for higher dimensions, we can use the canonical d-connection ${}^k\widehat{\mathbf{D}}$ and its nonholonomic deformations for equivalent reformulations of extra dimension gravity theories and as tools for generating constructing exact solutions. In particular, imposing necessary type constraints, it is possible to generate exact solutions of the Einstein equations for the Levi-Civita connection ${}^k\nabla$.

3 N-adapted Einstein Equations

In this section, we define the Riemannian, Ricci and Einstein tensors for the canonical d-connection ${}^k\widehat{\mathbf{D}}$ and metric ${}^k\mathbf{g}$ (14) and derive the corresponding gravitational field equations. We also formulate the general conditions when the Einstein tensor for ${}^k\widehat{\mathbf{D}}$ is equal to that for ${}^k\nabla$.

3.1 Curvature of the Canonical d-connection

As for any linear connection, we can introduce:

Definition 3.1 The curvature of $\widehat{\mathbf{D}}$ is a 2-form $\widehat{\mathcal{R}} \doteqdot \widehat{\mathbf{D}}\widehat{\mathbf{\Gamma}} = d\widehat{\mathbf{\Gamma}} - \widehat{\mathbf{\Gamma}} \wedge \widehat{\mathbf{\Gamma}}$.

In explicit form, the N-adapted coefficients can be computed using the 1-form (16),

$$\widehat{\mathcal{R}}^{k_\alpha}{}_{k_\beta} \doteqdot \widehat{\mathbf{D}}\widehat{\mathbf{\Gamma}}^{k_\alpha}{}_{k_\beta} = d\widehat{\mathbf{\Gamma}}^{k_\alpha}{}_{k_\beta} - \widehat{\Gamma}^{k_\gamma}{}_{k_\beta} \wedge \widehat{\mathbf{\Gamma}}^{k_\alpha}{}_{k_\gamma} = \widehat{\mathbf{R}}^{k_\alpha}{}_{k_\beta\, k_\gamma\, k_\tau}\mathbf{e}^{k_\gamma} \wedge \mathbf{e}^{k_\tau}, \tag{23}$$

The N-adapted coefficients of curvature are parametrized in the form:

$$\begin{aligned}
\widehat{\mathbf{R}}^{k_\alpha}{}_{k_\beta\, k_\gamma\, k_\tau} = \langle \widehat{R}^{\alpha}{}_{\beta\gamma\tau} &= \{\widehat{R}^{i}{}_{hjk}, \widehat{R}^{a}{}_{bjk}, \widehat{R}^{i}{}_{jka}, \widehat{R}^{c}{}_{bka}, \widehat{R}^{i}{}_{jbc}, \widehat{R}^{a}{}_{bcd}\};\\
\widehat{R}^{1_\alpha}{}_{1_\beta\, 1_\gamma\, 1_\tau} &= \{\widehat{R}^{\alpha}{}_{\beta\gamma\tau}, \widehat{R}^{1_a}{}_{1_b\gamma\tau}, \widehat{R}^{\alpha}{}_{\beta\gamma\, 1_a}, \widehat{R}^{1_c}{}_{1_b\gamma\, 1_a}, \widehat{R}^{\alpha}{}_{\beta\, 1_b\, 1_c}, \widehat{R}^{1_a}{}_{1_b\, 1_c\, 1_d}\};\\
\widehat{R}^{2_\alpha}{}_{2_\beta\, 2_\gamma\, 2_\tau} &= \{\widehat{R}^{1_\alpha}{}_{1_\beta\, 1_\gamma\, 1_\tau}, \widehat{R}^{2_a}{}_{2_b\, 1_\gamma\, 1_\tau}, \widehat{R}^{1_\alpha}{}_{1_\beta\, 1_\gamma\, 2_a}, \widehat{R}^{2_c}{}_{2_b\, 1_\gamma\, 2_a},\\
&\qquad \widehat{R}^{1_\alpha}{}_{1_\beta\, 2_b\, 2_c}, \widehat{R}^{2_a}{}_{2_b\, 2_c\, 2_d}\};\\
&\dots\\
\widehat{R}^{k_\alpha}{}_{k_\beta\, k_\gamma\, k_\tau} &= \{\widehat{R}^{k-1_\alpha}{}_{k-1_\beta\, k-1_\gamma\, k-1_\tau}, \widehat{R}^{k_a}{}_{k_b\, k-1_\gamma\, k-1_\tau}, \widehat{R}^{k-1_\alpha}{}_{k-1_\beta\, k-1_\gamma\, k_a},\\
&\qquad \widehat{R}^{k_c}{}_{k_b\, k-1_\gamma\, k_a}, \widehat{R}^{k-1_\alpha}{}_{k-1_\beta\, k_b\, k_c}, \widehat{R}^{k_a}{}_{k_b\, k_c\, k_d}\}\rangle,
\end{aligned} \tag{24}$$

where the values of such coefficients are provided in Appendix A, see Theorem A.1 and formulas (51).

Definition 3.2 The Ricci tensor $\mathrm{Ric}(\widehat{\mathbf{D}}) = \{\widehat{\mathbf{R}}_{\alpha\beta}\}$ of a canonical d-connection $\widehat{\mathbf{D}}$ is defined by contracting respectively the N-adapted coefficients of $\widehat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\delta}$ (23), when $\widehat{\mathbf{R}}_{\alpha\beta} \doteqdot \widehat{\mathbf{R}}^{\tau}{}_{\alpha\beta\tau}$.

We formulate:

Corollary 3.1 *The Ricci tensor of $\widehat{\mathbf{D}}$ is characterized by N-adapted coefficients*

$$\begin{aligned}
\widehat{\mathbf{R}}_{k_\alpha\, k_\beta} = \{&\widehat{R}_{ij}, \widehat{R}_{ia},\ \widehat{R}_{ai},\ \widehat{R}_{ab}; \widehat{R}_{\alpha\beta}, \widehat{R}_{\alpha\, 1_a},\ \widehat{R}_{1_a\beta},\ \widehat{R}_{1_a\, 1_b};\\
&\widehat{R}_{1_\alpha\, 1_\beta}, \widehat{R}_{1_\alpha\, 2_a},\ \widehat{R}_{2_a\, 1_\beta},\ \widehat{R}_{2_a\, 2_b}; \dots;\\
&\widehat{R}_{k-1_\alpha\, k-1_\beta}, \widehat{R}_{k-1_\alpha\, k_a},\ \widehat{R}_{k_a\, k-1_\beta},\ \widehat{R}_{k_a\, k_b}\},
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
&\widehat{R}_{ij} \doteqdot \widehat{R}^{k}{}_{ijk}, \qquad \widehat{R}_{ia} \doteqdot -\widehat{R}^{k}{}_{ika}, \qquad \widehat{R}_{ai} \doteqdot \widehat{R}^{b}{}_{aib}, \qquad \widehat{R}_{ab} \doteqdot \widehat{R}^{c}{}_{abc};\\
&\widehat{R}_{\alpha\beta} \doteqdot \widehat{R}^{\gamma}{}_{\alpha\beta\gamma}, \qquad \widehat{R}_{\alpha\, 1_a} \doteqdot -\widehat{R}^{\gamma}{}_{\alpha\gamma\, 1_a}, \qquad \widehat{R}_{1_a\alpha} \doteqdot \widehat{R}^{1_b}{}_{1_a\alpha\, 1_b},\\
&\qquad \widehat{R}_{1_a\, 1_b} \doteqdot \widehat{R}^{1_c}{}_{1_a\, 1_b\, 1_c};\\
&\widehat{R}_{1_\alpha\, 1_\beta} \doteqdot \widehat{R}^{1_\gamma}{}_{1_\alpha\, 1_\beta\, 1_\gamma}, \qquad \widehat{R}_{1_\alpha\, 2_a} \doteqdot -\widehat{R}^{1_\gamma}{}_{1_\alpha\, 1_\gamma\, 2_a},\\
&\widehat{R}_{2_a\, 1_\alpha} \doteqdot \widehat{R}^{2_b}{}_{2_a\, 1_\alpha\, 2_b}, \qquad \widehat{R}_{2_a\, 2_b} \doteqdot \widehat{R}^{2_c}{}_{2_a\, 2_b\, 2_c};\\
&\qquad \dots
\end{aligned} \tag{26}$$

$$\widehat{R}_{{}^{k-1}\alpha\ {}^{k-1}\beta} \doteqdot \widehat{R}^{{}^{k-1}\gamma}{}_{{}^{k-1}\alpha\ {}^{k-1}\beta\ {}^{k-1}\gamma}, \qquad \widehat{R}_{{}^{k-1}\alpha\ {}^{k}a} \doteqdot -\widehat{R}^{{}^{k-1}\gamma}{}_{{}^{k-1}\alpha\ {}^{k-1}\gamma\ {}^{k}a},$$
$$\widehat{R}_{{}^{k}a\ {}^{k-1}\alpha} \doteqdot \widehat{R}^{{}^{k}b}{}_{{}^{k}a\ {}^{k-1}\alpha\ {}^{k}b}, \qquad \widehat{R}_{{}^{k}a\ {}^{k}b} \doteqdot \widehat{R}^{{}^{k}c}{}_{{}^{k}a\ {}^{k}b\ {}^{k}c}.$$

Proof The formulas (26) follow from contractions of (24). To compute the N-adapted coefficients of the Ricci tensor $\mathrm{Ric}(\widehat{\mathbf{D}})$ for metric ${}^{k}\mathbf{g}$ (14) we have to construct correspondingly the formulas (51). □

Definition 3.3 The scalar curvature ${}^{s}\widehat{R}$ of $\widehat{\mathbf{D}}$ is by definition

$$\begin{aligned} {}^{s}\widehat{R} &\doteqdot \mathbf{g}^{{}^{k}\alpha\ {}^{k}\beta}\widehat{\mathbf{R}}_{{}^{k}\alpha\ {}^{k}\beta} \\ &= g^{ij}\widehat{R}_{ij} + h^{ab}\widehat{R}_{ab} + h^{{}^{1}a\ {}^{1}b}\widehat{R}_{{}^{1}a\ {}^{1}b} + \cdots + h^{{}^{k}a\ {}^{k}b}\widehat{R}_{{}^{k}a\ {}^{k}b}. \end{aligned} \tag{27}$$

Using values (26) and (27), we can compute the Einstein tensor $\widehat{\mathbf{E}}_{{}^{k}\alpha\ {}^{k}\beta}$ of $\widehat{\mathbf{D}}$,

$$\widehat{\mathbf{E}}_{{}^{k}\alpha\ {}^{k}\beta} \doteqdot \widehat{\mathbf{R}}_{{}^{k}\alpha\ {}^{k}\beta} - \frac{1}{2}\mathbf{g}_{{}^{k}\alpha\ {}^{k}\beta}\ {}^{s}\widehat{R}. \tag{28}$$

In explicit form, for N-adapted coefficients, we get a proof for

Corollary 3.2 *The Einstein tensor* $\widehat{\mathbf{E}}_{{}^{k}\alpha\ {}^{k}\beta}$ *splits into h- and* ${}^{k}v$*-components* $\widehat{\mathbf{E}}_{{}^{k}\alpha\ {}^{k}\beta} \doteqdot \{\widehat{E}_{ij} = \widehat{R}_{ij} - \frac{1}{2}g_{ij}{}^{s}\widehat{R}, \widehat{E}_{ia} = \widehat{R}_{ia}, \widehat{E}_{ai} = \widehat{R}_{ai}, \ \widehat{E}_{ab} = \widehat{R}_{ab} - \frac{1}{2}h_{ab}\ {}^{s}\widehat{R};\ \widehat{E}_{\alpha\ {}^{1}a} = \widehat{R}_{\alpha\ {}^{1}a},\ \widehat{E}_{{}^{1}a\beta} = \widehat{R}_{{}^{1}a\beta},\ \ \widehat{E}_{{}^{1}a\ {}^{1}b} = \widehat{R}_{{}^{1}a\ {}^{1}b} - \frac{1}{2}h_{{}^{1}a\ {}^{1}b}\ {}^{s}\widehat{R};\ \widehat{E}_{{}^{1}\alpha\ {}^{2}a} = \widehat{R}_{{}^{1}\alpha\ {}^{2}a},\ \ \widehat{E}_{{}^{2}a\ {}^{1}\beta} = \ \widehat{R}_{{}^{2}a\ {}^{1}\beta},\ \ \widehat{E}_{{}^{2}a\ {}^{2}b} = \widehat{R}_{{}^{2}a\ {}^{2}b} - \frac{1}{2}h_{{}^{2}a\ {}^{2}b}\ {}^{s}\widehat{R};\ \widehat{E}_{{}^{k-1}\alpha\ {}^{k}a} = \widehat{R}_{{}^{k-1}\alpha\ {}^{k}a},\ \widehat{E}_{{}^{k}a\ {}^{k-1}\beta} = \widehat{R}_{{}^{k}a\ {}^{k-1}\beta}, \widehat{E}_{{}^{k}a\ {}^{k}b} = \widehat{R}_{{}^{k}a\ {}^{k}b} - \frac{1}{2}h_{{}^{k}a\ {}^{k}b}\ {}^{s}\widehat{R}\}$.

In different theories of (string/brane/gauge etc.) gravity, we can consider nonholonomically modified gravitational field equations

$$\widehat{\mathbf{E}}_{{}^{k}\alpha\ {}^{k}\beta} = \varkappa\widehat{\mathbf{T}}_{{}^{k}\alpha\ {}^{k}\beta}, \tag{29}$$

for a source $\widehat{\mathbf{T}}_{{}^{k}\alpha\ {}^{k}\beta}$ defined by certain classical or quantum corrections and/or constraints on dynamics of fields to usual energy-momentum tensors. Such equations are not equivalent, in general, to the usual Einstein equations (2) for the Levi-Civita connection ${}^{k}\nabla$.[10]

Condition 3.1 *A class of metrics* ${}^{k}\mathbf{g}$ (14) *defining solutions of the gravitational field equations the canonical d-connection* (29) *are also solutions for the Einstein equations for the Levi-Civita connection* (2), *if with respect to certain N-adapted frames* (10) *and* (11) *there are satisfied the conditions*

$$\begin{aligned} &\widehat{C}^{i}_{jb} = 0, \quad \Omega^{a}{}_{ji} = 0, \quad \widehat{T}^{c}_{ja} = 0; \quad \widehat{C}^{\alpha}_{\beta\ {}^{1}b} = 0, \quad \Omega^{{}^{1}a}_{\beta\alpha} = 0, \quad \widehat{T}_{\beta}{}^{{}^{1}c}{}_{{}^{1}a} = 0; \\ &\widehat{C}^{{}^{1}\alpha}_{{}^{1}\beta\ {}^{2}b} = 0, \quad \Omega^{{}^{2}a}{}_{{}^{1}\beta\ {}^{1}\alpha} = 0, \quad \widehat{T}^{{}^{2}c}_{{}^{1}\beta\ {}^{2}a} = 0; \ldots; \\ &\widehat{C}^{{}^{k-1}\alpha}_{{}^{k-1}\beta\ {}^{k}b} = 0, \quad \Omega^{{}^{k}a}{}_{{}^{k-1}\beta\ {}^{k-1}\alpha} = 0, \quad \widehat{T}^{{}^{k}c}_{{}^{k-1}\beta\ {}^{k}a} = 0; \end{aligned} \tag{30}$$

[10] As we noted in Refs. [4, 37–39], an equivalence of both types of filed equations would be possible, for instance, if we introduce a generalized source $\widehat{\mathbf{T}}_{{}^{k}\beta\ {}^{k}\delta}$ containing contributions of the distortion tensor (22).

and $\varkappa\widehat{\mathbf{T}}_{^k\alpha\,^k\beta}$ includes the energy-momentum tensor for matter field in usual gravity and distortions of the Einstein tensor determined by distortions of linear connections.

Proof We can see that both the torsion (20) and distortion tensor, see formulas (22), became zero if the conditions (30) are satisfied. In such a case, the distortion relations (21) transform into $\Gamma^{^k\gamma}_{\ ^k\alpha\,^k\beta}=\widehat{\mathbf{\Gamma}}^{^k\gamma}_{\ ^k\alpha\,^k\beta}$ (even, in general, $\widehat{\mathbf{D}}\neq\nabla$).[11] Even such additional constraints are imposed, the geometric constructions are with nonholonomic variables because the anholonomy coefficients are not obligatory zero (for instance, $w^b_{ia}=\partial_a N^b_i$ etc, see formulas (12)). □

3.2 The System of N-adapted Einstein Equations

The goal of this work is to prove that we can solve in a very general form any system of gravitational filed equations in high dimensional gravity, for instance, for the canonical d-connection and/or the Levi-Civita connection if such equations can be written as a variant of (29),

$$\widehat{\mathbf{R}}^{^k\alpha}_{\ \ ^k\beta}=\Upsilon^{^k\alpha}_{\ \ ^k\beta}, \tag{31}$$

with a general source parametrize in the form (4), $\Upsilon^{^k\alpha}_{\ \ ^k\beta}=\mathrm{diag}[\Upsilon_{^k\gamma}]$, including possible contributions from energy-momentum and/or distortion tensors.

Let us denote partial derivatives in the form

$$\partial_2=\partial/\partial x^2,\dots,\partial_v=\partial/\partial v,\partial_{^kv}=\partial/\partial\,^kv,\dots,\partial_\alpha=\partial/\partial u^\alpha,\dots,\partial_{^k\alpha}=\partial/\partial u^{^k\alpha}.$$

Theorem 3.1 *The gravitational field equations (31) constructed for $^k\widehat{\mathbf{D}}=\{\widehat{\mathbf{\Gamma}}^{^k\gamma}_{\ ^k\alpha\,^k\beta}\}$ with coefficients (17) and computed for a metric $^k\mathbf{g}=\{\mathbf{g}_{^k\beta\,^k\gamma}\}$ (14) with coefficients (15) are equivalent to this system of partial differential equations*:

$$\begin{aligned}\widehat{R}^2_2=\widehat{R}^3_3=\frac{1}{2g_2g_3}\bigg[&\frac{\partial_2g_2\cdot\partial_2g_3}{2g_2}+\frac{(\partial_2g_3)^2}{2g_3}-\partial_2{}^2g_3\\&+\frac{\partial_3g_2\cdot\partial_3g_3}{2g_3}+\frac{(\partial_3g_2)^2}{2g_2}-\partial_3{}^2g_2\bigg]=-\Upsilon_4(x^{\hat{i}}),\end{aligned}\tag{32}$$

$$\widehat{R}^4_4=\widehat{R}^5_5=\frac{\partial_vh_5}{2h_4h_5}\partial_v\left(\ln\left|\frac{\sqrt{|h_4h_5|}}{\partial_vh_5}\right|\right)=-\Upsilon_2(x^i,v),\tag{33}$$

$$\widehat{R}_{4i}=-w_i\frac{\beta}{2h_4}-\frac{\alpha_i}{2h_4}=0,\tag{34}$$

$$\widehat{R}_{5i}=-\frac{h_5}{2h_4}[\partial_v{}^2n_i+\gamma\partial_vn_i]=0,\tag{35}$$

$$\widehat{R}^6_6=\widehat{R}^7_7=\frac{\partial_{^1v}h_7}{2h_6h_7}\partial_{^1v}\left(\ln\left|\frac{\sqrt{|h_6h_7|}}{\partial_{\,^1v}h_6}\right|\right)=-^1\Upsilon_2(u^\alpha,\ ^1v),$$

$$\widehat{R}_{6\mu}=-w_\mu\frac{^1\beta}{2h_6}-\frac{\alpha_\mu}{2h_6}=0,\qquad\widehat{R}_{7\mu}=-\frac{h_7}{2h_6}[\partial^2_{^1v}n_\mu+{}^1\gamma\partial_{^1v}n_\mu]=0,$$

[11] This is possible because the laws of transforms for d-connections, for the Levi-Civita connection and different types of tensors being adapted, or not, to a N-splitting (9) are very different.

$$\widehat{R}^{8}_{8} = \widehat{R}^{9}_{9} = \frac{\partial_{2_v} h_9}{2h_8 h_9}\partial_{2_v}\left(\ln\left|\frac{\sqrt{|h_8 h_9|}}{\partial_{2_v} h_8}\right|\right) = -{}^{2}\Upsilon_2(u^{{}^1\alpha},\ {}^2v),$$

$$\widehat{R}_{8\ {}^1\mu} = -w_{{}^1\mu}\frac{{}^2\beta}{2h_8} - \frac{\alpha_{{}^1\mu}}{2h_8} = 0,$$

$$\widehat{R}_{9\ {}^1\mu} = -\frac{h_9}{2h_8}[\partial^2_{2_v} n_{{}^1\mu} + {}^2\gamma\,\partial_{2_v} n_{{}^1\mu}] = 0,$$

$$\dots$$

$$\widehat{R}^{4+2k}_{4+2k} = \widehat{R}^{5+2k}_{5+2k} = \frac{\partial_{\ {}^k v} h_{5+2k}}{2h_{4+2k} h_{5+2k}}\partial_{\ {}^k v}\left(\ln\left|\frac{\sqrt{|h_{4+2k} h_{5+2k}|}}{\partial_{{}^k v} h_{4+2k}}\right|\right)$$

$$= -{}^{k}\Upsilon_2(u^{{}^{k-1}\alpha},\ {}^k v),$$

$$\widehat{R}_{4+2k\ {}^{k-1}\mu} = -w_{{}^{k-1}\mu}\frac{{}^k\beta}{2h_{4+2k}} - \frac{\alpha_{\ {}^{k-1}\mu}}{2h_{4+2k}} = 0,$$

$$\widehat{R}_{5+2k\ {}^{k-1}\mu} = -\frac{h_{5+2k}}{2h_{4+2k}}[\partial^2_{{}^k v} n_{{}^{k-1}\mu} + {}^k\gamma\,\partial_{{}^k v} n_{{}^{k-1}\mu}] = 0,$$

where, for $\partial_v h_4 \neq 0$ *and* $\partial_v h_5 \neq 0$; $\partial_{1_v} h_6 \neq 0$ *and* $\partial_{1_v} h_7 \neq 0$; $\partial_{2_v} h_8 \neq 0$ *and* $\partial_{2_v} h_9 \neq 0; \dots; \partial_{{}^k v} h_{4+2k} \neq 0$ *and* $\partial_{{}^k v} h_{5+2k} \neq 0$;[12]

$$\phi = \ln\left|\frac{\partial_v h_5}{\sqrt{|h_4 h_5|}}\right|, \qquad \alpha_i = \partial_v h_5 \cdot \partial_i \phi,$$

$$\beta = \partial_v h_4 \cdot \partial_v \phi, \qquad \gamma = \partial_v(\ln|h_5|^{3/2}/|h_4|); \tag{36}$$

$${}^1\phi = \ln\left|\frac{\partial_{1_v} h_7}{\sqrt{|h_6 h_7|}}\right|, \qquad \alpha_\mu = \partial_{1_v} h_7 \cdot \partial_\mu\ {}^1\phi,$$

$${}^1\beta = \partial_{1_v} h_6\ \cdot \partial_{1_v}\ {}^1\phi, \qquad {}^1\gamma = \partial_{1_v}\big(\ln|h_7|^{3/2}/|h_6|\big);$$

$${}^2\phi = \ln\left|\frac{\partial_{2_v} h_9}{\sqrt{|h_8 h_9|}}\right|, \qquad \alpha_{{}^1\mu} = \partial_{2_v} h_9\ \cdot \partial^2_{{}^1\mu}\phi,$$

$${}^2\beta = \partial_{2_v} h_8\ \cdot \partial_{2_v}\ {}^2\phi, \qquad {}^2\gamma = \partial_{2_v}\big(\ln|h_9|^{3/2}/|h_8|\big);$$

$$\dots$$

$${}^k\phi = \ln\left|\frac{\partial_{\ {}^k v} h_{5+2k}}{\sqrt{|h_{4+2k} h_{5+2k}|}}\right|, \qquad \alpha_{{}^{k-1}\mu} = \partial_{{}^k v} h_{5+2k}\ \cdot \partial_{{}^{k-1}\mu}\ {}^k\phi,$$

$${}^k\beta = \partial_{{}^k v} h_{4+2k} \cdot \partial_{{}^k v}\ {}^k\phi, \qquad {}^k\gamma = \partial_{{}^k v}\big(\ln|h_{5+2k}|^{3/2}/|h_{4+2k}|\big).$$

Proof of this theorem is sketched in Appendix B.

Finally, we emphasize that the system of equations constructed in Theorem 3.1 can be integrated in very general forms. For instance, for any given Υ_4 and Υ_4, (32) relates an un-known function $g_2(x^2,x^3)$ to a prescribed $g_3(x^2,x^3)$, or inversely. Equation (33) contains only derivatives on $y^4 = v$ and allows us to define $h_4(x^i,v)$ for a given $h_5(x^i,v)$, or inversely, for $h^*_{4,5} \neq 0$. Having defined h_4 and h_5, we can compute the coefficients (36),

[12] Solutions, for instance, with $\partial_v h_4 = 0$ and/or $\partial_v h_5 = 0$, should be analyzed as some special cases (for simplicity, we omit such considerations in this work).

which allows us to find w_i from algebraic equations (34) and to compute n_i by integrating two times on v as follow from (35). Similar properties hold true for equations on higher order shells.

4 General Solutions for Einstein Equations with Extra Dimensions

In this section, we show how general solutions of the gravitational field equations can be constructed in explicit form. There are three key steps: The first one is to generate exact solutions with Killing symmetries for the canonical d-connection. At the second one, we shall analyze the constraints selecting solutions for the Levi-Civita connections. The final (third) step will be in generalizing the constructions by eliminating Killing symmetries.

4.1 Exact Solutions with Killing Symmetries

We formulate for the Einstein equations for the canonical d-connection:

Theorem 4.1 *The general class of solutions of nonholonomic gravitational equations* (31) *with Killing symmetries on $e_{5+2k} = \partial/\partial y^{5+2k}$ is defined by ansatz of type* (5) *with ${}^k\omega^2 = 1$ and coefficients $g_{\hat{i}}, h_{{}^k a}, w_{{}^{k-1}\alpha}, n_{{}^{k-1}\alpha}$ computed for $k = 0, 1, 2, \ldots$ following formulas* (6).

Proof We sketch the proof giving more details for the shell $k = 0$ (higher order constructions being similar):

- The general solution of (32) can be written in the form $\varpi = g_{[0]} \exp[a_2 \widetilde{x}^2(x^2, x^3) + a_3 \widetilde{x}^3(x^2, x^3)]$, were $g_{[0]}, a_2$ and a_3 are some constants and the functions $\widetilde{x}^{2,3}(x^2, x^3)$ define any coordinate transforms $x^{2,3} \to \widetilde{x}^{2,3}$ for which the 2D line element becomes conformally flat, i.e.

  $$g_2(x^2, x^3)(dx^2)^2 + g_3(x^2, x^3)(dx^3)^2 \to \varpi(x^2, x^3)\big[(d\widetilde{x}^2)^2 + \epsilon(d\widetilde{x}^3)^2\big],$$

  where $\epsilon = \pm 1$ for a corresponding signature. It is convenient to write some partial derivatives, in brief, in the form $\partial_2 g = g^{\bullet}, \partial_3 g = g', \partial_4 g = g^*$. In coordinates $\widetilde{x}^{2,3}$, (32) transform into $\varpi(\ddot{\varpi} + \varpi'') - \dot{\varpi} - \varpi' = 2\varpi^2 \Upsilon_4(\tilde{x}^2, \tilde{x}^3)$ or

  $$\ddot{\psi} + \psi'' = 2\Upsilon_4(\tilde{x}^2, \tilde{x}^3), \tag{37}$$

  for $\psi = \ln|\varpi|$. The integrals of (37) depends on the source Υ_4. As a particular case we can consider that $\Upsilon_4 = 0$.
- For ${}^k\Upsilon_2(u^{{}^{k-1}\alpha}, {}^k v) = 0$, (33), and its higher shell analogs, relates two functions $h_{4+2k}(u^{{}^{k-1}\alpha}, {}^k v)$ and $h_{5+2k}(u^{{}^{k-1}\alpha}, {}^k v)$ following two possibilities:

  (a) to compute

  $$\begin{aligned}\sqrt{|h_{5+2k}|} &= {}_1h_{5+2k}(u^{{}^{k-1}\alpha}) + {}_2h_{5+2k}(u^{{}^{k-1}\alpha}) \\ &\quad \times \int \sqrt{|h_{4+2k}(u^{{}^{k-1}\alpha}, {}^k v)|}dv, \quad \text{for } \partial_{{}^k v} h_{4+2k}(u^{{}^{k-1}\alpha}, {}^k v) \neq 0; \\ &= {}_1h_{5+2k}(u^{{}^{k-1}\alpha}) + {}_2h_{5+2k}(u^{{}^{k-1}\alpha})^k v, \\ \text{for} \quad &\partial_{{}^k v} h_{4+2k}(u^{{}^{k-1}\alpha}, {}^k v) = 0,\end{aligned} \tag{38}$$

for some functions ${}_1h_{5+2k}(u^{k-1\alpha})$ and ${}_2h_{5+2k}(u^{k-1\alpha})$ stated by boundary conditions;

(b) or, inversely, to compute h_{4+2k} for respectively given h_{5+2k}, with $\partial_{^k v}h_{5+2k} \neq 0$,

$$\sqrt{|h_{4+2k}|} = {}_k^0h(u^{k-1\alpha})\ \partial_{^k v}\sqrt{|h_{5+2k}(u^{k-1\alpha},\ {}^kv)|}, \tag{39}$$

with ${}_k^0h(u^{k-1\alpha})$ given by boundary conditions. We note that the (33) with zero source is satisfied by arbitrary pairs of coefficients $h_{4+2k}(u^{k-1\alpha},\ {}^kv)$ and ${}_0h_{5+2k}(u^{k-1\alpha})$. Solutions with ${}^k\Upsilon_2 \neq 0$ can be found by ansatz of type

$$h_{5+2k}[{}^k\Upsilon_2] = h_{5+2k},\, h_4[\,{}^k\Upsilon_2] = \varsigma_{4+2k}(u^{k-1\alpha},\ {}^kv)h_{4+2k}, \tag{40}$$

where h_{4+2k} and h_{5+2k} are related by formula (38), or (39). Substituting (40), we obtain

$$\varsigma_{4+2k}(u^{k-1\alpha},\ {}^kv) = {}^0\varsigma_{4+2k}(u^{k-1\alpha}) - \int {}^k\Upsilon_2 \frac{h_{4+2k}h_{5+2k}}{4\partial_{^k v}h_{5+2k}} d^kv, \tag{41}$$

where ${}^0\varsigma_{4+2k}(u^{k-1\alpha})$ are arbitrary functions.

- The exact solutions of (34) for $\beta \neq 0$ are defined from an algebraic equation, $w_i\beta + \alpha_i = 0$, where the coefficients β and α_i are computed as in formulas (36) by using the solutions for (32) and (33). The general solution is

$$w_k = \partial_k \ln[\sqrt{|h_4h_5|}/|h_5^*|]/\partial_v \ln[\sqrt{|h_4h_5|}/|h_5^*|], \tag{42}$$

with $\partial_v = \partial/\partial v$ and $h_5^* \neq 0$. If $h_5^* = 0$, or even $h_5^* \neq 0$ but $\beta = 0$, the coefficients w_k could be arbitrary functions on (x^i, v). For the vacuum Einstein equations this is a degenerated case imposing the compatibility conditions $\beta = \alpha_i = 0$, which are satisfied, for instance, if the h_4 and h_5 are related as in the formula (39) but with $h_{[0]}(x^i) = \text{const}$.
- Having defined h_4 and h_5 and computed γ from (36), we can solve (35) by integrating on variable "v" the equation $n_i^{**} + \gamma n_i^* = 0$. The exact solution is

$$\begin{aligned} n_k &= n_{k[1]}(x^i) + n_{k[2]}(x^i)\int [h_4/(\sqrt{|h_5|})^3]dv, \qquad h_5^* \neq 0; \\ &= n_{k[1]}(x^i) + n_{k[2]}(x^i)\int h_4 dv, \qquad h_5^* = 0; \\ &= n_{k[1]}(x^i) + n_{k[2]}(x^i)\int [1/(\sqrt{|h_5|})^3]dv, \qquad h_4^* = 0, \end{aligned} \tag{43}$$

for some functions $n_{k[1,2]}(x^i)$ stated by boundary conditions.
- The generating and integration formulas in higher order formulas (40), (41), (42), (43) etc. are redefined in a form as it was considered for $k = 0$ in review articles [1, 4] which result in formulas (6) for ${}^k\omega^2 = 1$. □

We note that the solutions constructed in Theorem 4.1 are very general ones and contain as particular cases all known exact solutions for (non) holonomic Einstein spaces with Killing symmetries. They also can be generalized to include arbitrary finite sets of parameters, see Ref. [1].

Corollary 4.1 *An ansatz* (5) *with* ${}^k\omega^2 = 1$ *and coefficients* $g_{\hat{i}}, h_{^k a}, w_{^{k-1}\beta}, n_{^{k-1}\beta}$ *computed following formulas* (6) *define solutions with Killing symmetries on* $e_{5+2k} = \partial/\partial y^{5+2k}$ *of the*

Einstein equations (3) *for the Levi-Civita connection* $\Gamma^{^k\gamma}_{\ ^k\alpha\ ^k\beta}$ *if the coefficients of metric are subjected additionally to the conditions* (8).

Proof By straightforward computations for ansatz (14) with coefficients (15), we get that the conditions (30) resulting in $\Gamma^{^k\gamma}_{\ ^k\alpha\ ^k\beta} = \widehat{\Gamma}^{^k\gamma}_{\ ^k\alpha\ ^k\beta}$ are just those written as (8). For such ansatz, one holds the conditions (54) and the N-connection and torsion coefficients vanish, i. e. the values (52) and (55) became zero. We get nonholonomic configurations for the Levi-Civita connection with nontrivial anholonomy coefficients (12). □

We conclude that in order to generate exact solutions with Killing symmetries in Einstein gravity and its higher order generalizations, we should consider N-adapted frames and nonholonomic deformations of the Levi-Civita connection to an auxiliary metric compatible d-connection (for instance, to the canonical d-connection, $^k\widehat{\mathbf{D}}$), when the corresponding system of nonholonomic gravitational field equations (32)–(35) can be integrated in general form. Subjecting the integral variety of such solutions to additional constraints of type (30), i.e. imposing the conditions (8) to the coefficients of metrics, we may construct new classes of exact solutions of Einstein equations for the Levi-Civita connection $^k\nabla$.

4.2 General Non-Killing Solutions

Our final aim is to consider general classes of solutions of the nonholonomic gravitational field equations (31), and (for more particular cases), of Einstein equations (3) metrics depending on all coordinates $u^{^k\alpha} = (x^i, y^{^ka})$, i.e. the solutions will be without Killing symmetries.

Let us introduce some nontrivial multiples $^k\omega^2(u^{^k\alpha})$ before coefficients $h_{^ka}$ parametrized in the form (15) and defining solutions with Killing symmetries. We get an ansatz

$$\begin{aligned} {}^k_\omega\mathbf{g} &= \epsilon_1 e^1 \otimes e^1 + g_{\widehat{j}}(x^{\widehat{k}}) e^{\widehat{j}} \otimes e^{\widehat{j}} + \omega^2(x^i, y^a) h_a(x^i, v) \mathbf{e}^a \otimes \mathbf{e}^a \\ &\quad + {}^1\omega^2(u^{^1\alpha}) h_{^1a\ ^1b}(u^\alpha,\ {}^1v) \mathbf{e}^{^1a} \otimes \mathbf{e}^{^1b} \\ &\quad + {}^2\omega^2(u^{^2\alpha}) h_{^2a\ ^2b}(u^{^1\alpha},\ {}^2v) \mathbf{e}^{^2a} \otimes \mathbf{e}^{^2b} + \cdots \\ &\quad + {}^k\omega^2(u^{^k\alpha})\ h_{^ka\ ^kb}(u^{^{k-1}\alpha},\ {}^kv) \mathbf{e}^{^ka} \otimes \mathbf{e}^{^kb}, \end{aligned} \tag{44}$$

where the N-adapted basis $\mathbf{e}^{^ka}$ (and N-connection) are the same as in (14). Under such noholonomic conformal transform[13] (defined by generating functions $^2\omega^2(u^{\ ^k\alpha})$) of metric, $^k\mathbf{g} \rightarrow {}_{^k\omega}\mathbf{g}$, the canonical d-connection deforms as $\widehat{\mathbf{\Gamma}}^{^k\gamma}_{\ ^k\alpha\ ^k\beta} \rightarrow {}_{^k\omega}\widehat{\mathbf{\Gamma}}^{^k\gamma}_{\ ^k\alpha\ ^k\beta}$, where ${}_{^k\omega}\widehat{\mathbf{\Gamma}}^{^k\gamma}_{\ ^k\alpha\ ^k\beta} = (\widehat{L}^i_{jk}, {}_\omega\widehat{L}^a_{bk}, \widehat{C}^i_{jc}, {}_\omega\widehat{C}^a_{bc}; {}_{^1\omega}\widehat{L}^{^1a}_{^1b\alpha}, {}_\omega\widehat{C}^\alpha_{\beta\ ^1c}, {}_{^1\omega}\widehat{C}^{^1a}_{^1b\ ^1c}; \ldots;\ {}_{^k\omega}\widehat{L}^{^ka}_{^kb\ ^{k-1}\alpha}, {}_{^{k-1}\omega}\widehat{C}^{^{k-1}\alpha}_{^{k-1}\beta\ ^kc}, {}_{^k\omega}\widehat{C}^{^ka}_{^kb\ ^kc})$,

$$\begin{aligned} \text{for}\quad \widehat{C}^i_{jc} = {}_\omega\widehat{C}^\alpha_{\beta^1c} = \cdots = {}_{^{k-1}\omega}\widehat{C}^{^{k-1}\alpha}_{^{k-1}\beta\ ^kc} = 0,\quad {}_{^k\omega}\widehat{L}^{^ka}_{^kb\ ^{k-1}\alpha} \\ = \widehat{L}^{^ka}_{^kb\ ^{k-1}\alpha} + {}^z_{^k\omega}\widehat{L}^{^ka}_{^kb\ ^{k-1}\alpha}, {}_{^k\omega}\widehat{C}^{^ka}_{^kb\ ^kc} = \widehat{C}^{^ka}_{^kb\ ^kc} + {}^z_{^k\omega}\widehat{C}^{^ka}_{^kb\ ^kc}, \end{aligned}$$

[13] We use the term “nonholonomic” because such transforms/deformations are adapted to a N-splitting stated by a prescribed nonholonomic distribution on a corresponding high dimension spacetime.

$$\text{with}\quad {}_{^k\omega}^{z}\widehat{L}^{k_a}_{k_b\ k-1_\alpha} = \frac{1}{2\ {}^k\omega^2} h^{k_a\ k_c}[h_{k_b\ k_c}\mathbf{e}_{\ k-1_\beta}({}^k\omega^2) - h_{k_b\ k_c}N^{k_d}_{k-1_\beta}\partial_{k_d}({}^k\omega^2)] = \delta^{k_a}_{\ k_b}\mathbf{e}_{k-1_\beta}\ln|{}^k\omega|; \tag{45}$$

$${}_{^k\omega}^{z}\widehat{C}^{k_a}_{k_b\ k_c} = \left(\delta^{k_a}_{\ k_b}\partial_{k_c} + \delta^{k_a}_{\ k_c}\partial_{k_b} - h_{k_b\ k_e}h^{k_a\ k_e}\partial_{\ k_e}\right)\ln|{}^k\omega|, \tag{46}$$

are computed by introducing coefficients of ${}^k_\omega\mathbf{g}$ (44) into (17).

Proposition 4.1 *For nonholonomic N-adapted transforms ${}^k\mathbf{g}$ (14) $\rightarrow {}_{^k\omega}\mathbf{g}$ (44) with shell coefficients ${}^k\omega$ satisfying respectively the conditions $\mathbf{e}_{k-1_\beta}({}^k\omega) = 0$, the Ricci tensor transform $\widehat{\mathbf{R}}_{k_\alpha\ k_\beta}$ (25) $\rightarrow {}_{^k\omega}\widehat{\mathbf{R}}_{k_\alpha\ k_\beta}$, where*

$$\begin{aligned}{}_{^k\omega}\widehat{\mathbf{R}}_{k_\alpha\ k_\beta} = \{&\widehat{R}_{ij}, \widehat{R}_{ia}, \ \widehat{R}_{ai}, {}_\omega\widehat{R}_{ab}; {}_\omega\widehat{R}_{\alpha\beta}, \widehat{R}_{\alpha^1 a}, \widehat{R}_{1_a\beta}, {}_{1_\omega}\widehat{R}_{1_a\ 1_b};\\ &{}_{1_\omega}\widehat{R}_{1_\alpha\ 1_\beta}, \widehat{R}_{1_\alpha\ 2_a}, \widehat{R}_{2_a\ 1_\beta}, {}_{2_\omega}\widehat{R}_{2_a\ 2_b}; \dots;\\ &{}_{k-1_\omega}\widehat{R}_{k-1_\alpha\ k-1_\beta}, \widehat{R}_{k-1_\alpha\ k_a}, \ \widehat{R}_{k_a\ k-1_\beta}, {}_{k_\omega}\widehat{R}_{k_a\ k_b}\},\end{aligned} \tag{47}$$

with ${}_{k-1_\omega}\widehat{R}_{k-1_\alpha\ k-1_\beta}$ computed recurrently using $\widehat{R}_{k-1_\alpha\ k-1_\beta}$ and ${}_{k_\omega}\widehat{R}_{k_a\ k_b} = \widehat{R}_{k_a\ k_b} + {}^{z}_{k_\omega}\widehat{R}_{k_a\ k_b}$, where the deformation tensor ${}^{z}_{k_\omega}\widehat{R}_{k_a\ k_b}$ is given by formula

$$\begin{aligned}{}^{z}_{k_\omega}\widehat{R}_{k_a\ k_b} = {}&(2 - {}^km)\widehat{D}_{k_a}\widehat{D}_{k_b}\ln|{}^k\omega| - h_{k_a\ k_b}h^{k_c\ k_d}\\ &\times \widehat{D}_{k_c}\widehat{D}_{k_d}\ln|{}^k\omega| - (2 - {}^km)(\widehat{D}_{k_a}\ln|\ {}^k\omega|)\widehat{D}_{k_b}\ln|\ {}^k\omega|\\ &+ (2 - {}^km)h_{k_a\ k_b}h^{k_c\ k_d}(\widehat{D}_{k_c}\ln|{}^k\omega|)\widehat{D}_{\ k_d}\ln|{}^k\omega|.\end{aligned} \tag{48}$$

Proof It follows from an explicit computation of N-adapted coefficients of (47) taking into account the deformation relations (45) and (46) when the condition $\mathbf{e}_{k-1_\beta}({}^k\omega) = 0$, which is just (7) from the Main Theorem 1.1. Working with shell coordinates y^{k_a}, the formulas for curvature and Ricci tensors are the same as on usual (pseudo) Riemannian spaces for the Levi-Civita connection, when coordinates of type x^i and y^{k-1_a} can be considered as some parameters. For "pure vertical" components, we can apply usual formulas for conformal transforms, like (46) and (48) outlined, for instance, in Appendix D of monograph [40]. □

Remark 4.1 (1) There are two reasons to consider two dimensional shells with ${}^km = 2$: The first one is that this results in field equations of type (33) which can be integrated in general form (it is a problem, at least technically, to find exact solutions for ${}^km > 2$). The second one is that from (48) we get ${}^{z}_{k_\omega}\widehat{R}^{k_a}_{\ k_b} = \delta^{k_a}_{\ k_b}\ {}^k\widehat{\square}\ln|{}^k\omega|$, with ${}^k\widehat{\square} \doteqdot h^{k_c\ k_d}\widehat{D}_{k_c}\widehat{D}_{k_d}$ being a shell type d'Alambert operator defined by the canonical d-connection.

(2) We can impose additionally the conditions

$${}^k\widehat{\square}\ln|{}^k\omega| = 0, \tag{49}$$

or to include such terms in sources (4), redefining the nonholonomic distributions to have

$${}^k\Upsilon_2(u^{k-1_\alpha},\ {}^kv) = {}^k_\omega\Upsilon_2(u^{k_\alpha}) - {}^k\widehat{\square}\ln|{}^k\omega(u^{k_\alpha})|, \tag{50}$$

for some well defined ${}^k_\omega\Upsilon_2(u^{k_\alpha})$ when formulas of type (41) can be computed. The conditions (49) or (50) can be selected also by corresponding integration functions, for instance, in (39), (42) and/or (43) and/or their higher shell analogs.

(3) Solutions with ${}^{k}m > 2$ can be with different topologies and generalized nonholonomic conformal symmetries. Locally such constructions may be performed in a simplest way by considering formulas only with ${}^{k}m = 2$ by increasing the number of "formal" shells.

As a result, we get the proof of

Lemma 4.1 *Any metric parametrized in the form* (44) *with coefficients depending on all variables on a* (*pseudo*) *Riemannian manifold* ${}^{k}\mathbf{V}$ ($\dim {}^{k}\mathbf{V} = 3$, *or* $2, +2k$; *with* $k = 0, 1, 2, \ldots$ *two dimensional shells*) *defines a "non-Killing" solution of the Einstein equations for the canonical d-connection* ${}^{k}\widehat{\mathbf{D}}$ *if the coefficients are given by data* (31) *as solutions with Killing symmetries of* (32)–(35), *when the parameters of nonholonomic conformal deformations* ${}^{k}\omega(u^{{}^{k}\alpha})$ *are chosen as generating functions satisfying the conditions* $\mathbf{e}_{{}^{k-1}\beta}({}^{k}\omega) = 0$ *and*, *for instance*, ${}^{k}\widehat{\square} \ln |{}^{k}\omega| = 0$. *Imposing additional restrictions on integration functions as in Corollary* 4.1, *we get general solutions for the Levi-Civita connection* ${}^{k}\nabla$.

Finally, in this section we formulate:

Conclusion 4.1 (1) Summarizing the Theorems 2.1–4.1 and Lemma 4.1, we prove the Main Result stated in Theorem 1.1.

(2) The general solutions defined by the conditions of Theorem 1.1 (and related results) can be extended to include contributions of an arbitrary number of commutative and noncommutative parameters. This is possible following the constructions with Killing symmetries, in our case for metrics (14) provided in Ref. [1], which can be similarly reconsidered with higher order shells.

5 Summary and Discussion

This work was primarily motivated by the question if the Einstein equations can be integrated in very general forms, for generic off-diagonal metrics depending on all possible variables, in arbitrary dimensions. To the best of our knowledge, such a problem has not yet been addressed in mathematical and physical literature being known the high complexity of related systems of nonlinear partial differential equations. This is in spite of the fact that there were elaborated a number of analytic and numerical methods of constructing exact and approximate solutions and that various types of such solutions seem to be of crucial physical importance in modern astrophysics and cosmology. Here we note that the bulk of former derived solutions are for diagonalizabe metrics (by coordinate transforms), depending on one and/or two (in some exceptional cases, on three) variables, with compactified dimensions, imposed symmetries, boundary conditions etc.

In a series of works, see reviews of results in Refs. [1–3], one of the main our goals was to formulate a geometric method which would allow us to construct exact solutions of gravitational field equations. We applied the formalism of nonholonomic distributions with generating and integration functions, when some of them are subjected to additional conditions/constraints (written as certain types of first order partial equations, algebraic relations, symmetry conditions etc.). That allowed us to elaborate a general scheme for deriving exact solutions with one Killing vector symmetry and various types of parametric dependencies. Finally, the so-called anholonomic deformation method was developed for general "non-Killing" solutions in paper [5].

Following the anholonomic deformation method, we define some "more convenient" holonomic and nonholonomic variables (frames coefficients and coordinates), which for certain types well defined conditions transform the Einstein equations into exactly integrable systems of equations. The key idea is to use additionally some auxiliary linear connections correspondingly adapted to nonholonomic distributions. Surprisingly, in our approach, it was possible to reformulate (and, in general, to modify) the Einstein equations in such forms, when general integral varieties can be constructed. Subjecting the coefficients of such way defined solutions to additional constraints, we can determine some integral subvarieties for standard gravity theories and generalizations. Here we note that our auxiliary connection (the so-called, canonical distinguished connection, in brief, d-connection) is also metric compatible and uniquely defined by the metric coefficients. It contains a nonholonomically induced torsion but such a geometric object is completely different from that, for instance, in Einstein–Cartan/gauge/string theory. In our approach, we do not need any additional field equations because we work with torsion coefficients induced by certain off-diagonal coefficients of metric. All geometric constructions can be equivalently performed using the Levi-Civita connection or, alternatively, the canonical d-connection.

Of course, our findings should be considered only in a line of qualitative understanding of the concept of general exact solutions in Einstein and high dimensional gravity. For such generic nonlinear systems, it is not possible to formulate any general uniqueness and completeness criteria for solution if we do not introduce any additional suppositions on classes of generating functions, symmetries, horizons, singularities, asymptotic conditions etc. Only in some more special/restricted cases, we can provide certain physical meaning for such general classes of solutions; to put, for instance, the Cauchy problem, construct some evolution models, determine symmetries of interactions etc. Our constructions are general ones because "almost" any solution in gravity theories can be parametrization in such a form at least locally even very different classes of metrics and connections can be stated globally for different topologies, boundary conditions, with various types of horizons and singularities etc. It is not our aim to perform such studies in this article.

Finally, we emphasize that the bulk of exact solutions in gravity theories (in Einstein gravity and various supersymmetric/noncommutative sting, brane, gauge, Kaluza–Klein, Lagrange–Finsler, generalizations etc.) can be represented in a form similar to (5). In this paper, we do not analyze possible explicit symmetries and physical properties of such solutions. We consider such problems in our recent papers [20–24, 38, 41–43] and plan to provide further developments and applications in our future works.

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Appendix A: Coefficients of N-adapted Curvature

In this section, we outline some formulas which play an important role in finding systems of partial differential equations which are equivalent to the Einstein equations with nonholonomic variables of arbitrary dimensions, see details in Refs. [1–4]. The formulas for coefficients of curvature $\widehat{\mathcal{R}}$ of the canonical d-connection $\widehat{\mathbf{D}}$ are written with respect to N-adapted frames (10) and (11).

Theorem A.1 *The curvature $\widehat{\mathcal{R}}$ (23) of the canonical d-connection $\widehat{\mathbf{D}}$ computed with respect to N-adapted frames (10) and (11) is characterized by coefficients*

$$
\begin{aligned}
\widehat{R}^i_{hjk} &= e_k\widehat{L}^i{}_{hj} - e_j\widehat{L}^i{}_{hk} + \widehat{L}^m_{hj}\widehat{L}^i{}_{mk} - \widehat{L}^m_{hk}\widehat{L}^i{}_{mj} - \widehat{C}^i{}_{ha}\Omega^a{}_{kj},\\
\widehat{R}^a{}_{bjk} &= e_k\widehat{L}^a{}_{bj} - e_j\widehat{L}^a{}_{bk} + \widehat{L}^c{}_{bj}\widehat{L}^a{}_{ck} - \widehat{L}^c{}_{bk}\widehat{L}^a{}_{cj} - \widehat{C}^a{}_{bc}\Omega^c{}_{kj},\\
\widehat{R}^i{}_{jka} &= e_a\widehat{L}^i{}_{jk} - \widehat{D}_k\widehat{C}^i{}_{ja} + \widehat{C}^i{}_{jb}\widehat{T}^b_{ka},\\
\widehat{R}^c{}_{bka} &= e_a\widehat{L}^c{}_{bk} - D_k\widehat{C}^c{}_{ba} + \widehat{C}^c{}_{bd}\widehat{T}^c{}_{ka},\\
\widehat{R}^i{}_{jbc} &= e_c\widehat{C}^i{}_{jb} - e_b\widehat{C}^i{}_{jc} + \widehat{C}^h{}_{jb}\widehat{C}^i{}_{hc} - \widehat{C}^h{}_{jc}\widehat{C}^i{}_{hb},\\
\widehat{R}^a{}_{bcd} &= e_d\widehat{C}^a{}_{bc} - e_c\widehat{C}^a{}_{bd} + \widehat{C}^e{}_{bc}\widehat{C}^a{}_{ed} - \widehat{C}^e{}_{bd}\widehat{C}^a{}_{ec};
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
\widehat{R}^\alpha{}_{\tau\beta\gamma} &= e_\gamma\widehat{L}^\alpha{}_{\tau\beta} - e_\beta\widehat{L}^\alpha{}_{\tau\gamma} + \widehat{L}^\mu{}_{\tau\beta}\widehat{L}^\alpha{}_{\mu\gamma} - \widehat{L}^\mu{}_{\tau\gamma}\widehat{L}^\alpha{}_{\mu\beta} - \widehat{C}^\alpha{}_{\tau\ {}^1a}\Omega^{{}^1a}{}_{\gamma\beta},\\
\widehat{R}^{{}^1a}{}_{{}^1b\beta\gamma} &= e_\gamma\widehat{L}^{{}^1a}{}_{{}^1b\beta} - e_\beta\widehat{L}^{{}^1a}{}_{{}^1b\gamma} + \widehat{L}^{{}^1c}{}_{{}^1b\beta}\widehat{L}^{{}^1a}{}_{{}^1c\gamma} - \widehat{L}^{{}^1c}{}_{{}^1b\gamma}\widehat{L}^{{}^1a}{}_{{}^1c\beta} - \widehat{C}^{{}^1a}{}_{{}^1b\ {}^1c}\Omega^{{}^1c}{}_{\gamma\beta},\\
\widehat{R}^\alpha{}_{\beta\gamma\ {}^1a} &= e_{{}^1a}\widehat{L}^\alpha{}_{\beta\gamma} - \widehat{D}_\gamma\widehat{C}^\alpha{}_{\beta\ {}^1a} + \widehat{C}^\alpha{}_{\beta\ {}^1b}\widehat{T}^{{}^1b}_{\gamma\ {}^1a},\\
\widehat{R}^{{}^1c}{}_{{}^1b\gamma\ {}^1a} &= e_{{}^1a}\widehat{L}^{{}^1c}_{{}^1b\gamma} - \widehat{D}_\gamma\widehat{C}^{{}^1c}{}_{{}^1b\ {}^1a} + \widehat{C}^{{}^1c}_{{}^1\ b\ {}^1d}\widehat{T}^{{}^1c}{}_{\gamma\ {}^1a},\\
\widehat{R}^\alpha{}_{\beta\ {}^1b\ {}^1c} &= e_{{}^1c}\widehat{C}^\alpha{}_{\beta\ {}^1b} - e_{{}^1b}\widehat{C}^\alpha{}_{\beta\ {}^1c} + \widehat{C}^\mu{}_{\beta\ {}^1b}\widehat{C}^\alpha{}_{\mu\ {}^1c} - \widehat{C}^\mu{}_{\beta\ {}^1c}\widehat{C}^\alpha{}_{\mu\ {}^1b},\\
\widehat{R}^{{}^1a}{}_{{}^1b\ {}^1c\ {}^1d} &= e_{{}^1d}\widehat{C}^{{}^1a}{}_{{}^1b\ {}^1c} - e_{{}^1c}\widehat{C}^{{}^1a}{}_{{}^1b\ {}^1d}\\
&\quad + \widehat{C}^{{}^1e}{}_{{}^1b\ {}^1c}\widehat{C}^{{}^1a}{}_{{}^1e\ {}^1d} - \widehat{C}^{{}^1e}{}_{{}^1b\ {}^1d}\widehat{C}^{{}^1a}{}_{{}^1e\ {}^1c};\\
\widehat{R}^{{}^1\alpha}{}_{{}^1\tau\ {}^1\beta\ {}^1\gamma} &= e_{{}^1\gamma}\widehat{L}^{{}^1\alpha}{}_{{}^1\tau\ {}^1\beta} - e_{{}^1\beta}\widehat{L}^{{}^1\alpha}{}_{{}^1\tau\ {}^1\gamma} + \widehat{L}^{{}^1\mu}{}_{{}^1\tau\ {}^1\beta}\widehat{L}^{{}^1\alpha}{}_{{}^1\mu\ {}^1\gamma}\\
&\quad - \widehat{L}^{{}^1\mu}{}_{{}^1\tau\ {}^1\gamma}\widehat{L}^{{}^1\alpha}{}_{{}^1\mu\ {}^1\beta} - \widehat{C}^{{}^1\alpha}{}_{{}^1\tau\ {}^2a}\Omega^{{}^2a}{}_{{}^1\gamma\ {}^1\beta},\\
\widehat{R}^{{}^2a}{}_{{}^2b\ {}^1\beta\ {}^1\gamma} &= e_{{}^1\gamma}\widehat{L}^{{}^2a}_{{}^2b\ {}^1\beta} - e_{{}^1\beta}\widehat{L}^{{}^2a}{}_{{}^2b\ {}^1\gamma}\\
&\quad + \widehat{L}^{{}^2c}{}_{{}^2b\ {}^1\beta}\widehat{L}^{{}^2a}{}_{{}^2c\ {}^1\gamma} - \widehat{L}^{{}^2c}{}_{{}^2b\ {}^1\gamma}\widehat{L}^{{}^2a}{}_{{}^2c\ {}^1\beta} - \widehat{C}^{{}^2a}{}_{{}^2b\ {}^2c}\Omega^{{}^2c}{}_{{}^1\gamma\ {}^1\beta},\\
\widehat{R}^{{}^1\alpha}{}_{{}^1\beta\ {}^1\gamma\ {}^2a} &= e_{{}^2a}\widehat{L}^{{}^1\alpha}{}_{{}^1\beta\ {}^1\gamma} - \widehat{D}_{{}^1\gamma}\widehat{C}^{{}^1\alpha}{}_{{}^1\beta\ {}^2a} + \widehat{C}^{{}^1\alpha}{}_{{}^1\beta\ {}^2b}\widehat{T}^{{}^2b}{}_{{}^1\gamma\ {}^2a},\\
\widehat{R}^{{}^2c}{}_{{}^2b\ {}^1\gamma\ {}^2a} &= e_{{}^2a}\widehat{L}^{{}^2c}{}_{{}^2b\ {}^1\gamma} - \widehat{D}_{{}^1\gamma}\widehat{C}^{{}^2c}{}_{{}^2b\ {}^2a} + \widehat{C}^{{}^2c}_{{}^2\ b\ {}^2d}\widehat{T}^{{}^2c}{}_{{}^1\gamma\ {}^2a},\\
\widehat{R}^{{}^1\alpha}{}_{{}^1\beta\ {}^2b\ {}^2c} &= e_{{}^2c}\widehat{C}^\alpha{}_{{}^1\beta\ {}^2b} - e_{{}^2b}\widehat{C}^{{}^1\alpha}{}_{{}^1\beta\ {}^2c}\\
&\quad + \widehat{C}^{{}^1\mu}{}_{{}^1\beta\ {}^2b}\widehat{C}^{{}^1\alpha}{}_{{}^1\mu\ {}^2c} - \widehat{C}^{{}^1\mu}{}_{{}^1\beta\ {}^2c}\widehat{C}^{{}^1\alpha}{}_{{}^1\mu\ {}^2b},\\
\widehat{R}^{{}^2a}{}_{{}^2b\ {}^2c\ {}^2d} &= e_{{}^2d}\widehat{C}^{{}^2a}{}_{{}^2b\ {}^2c} - e_{\ {}^2c}\widehat{C}^{{}^2a}{}_{{}^2b\ {}^2d}\\
&\quad + \widehat{C}^{{}^2e}{}_{{}^2b\ {}^2c}\widehat{C}^{{}^2a}{}_{{}^2e\ {}^2d} - \widehat{C}^{{}^2e}{}_{{}^2b\ {}^2d}\widehat{C}^{{}^2a}{}_{{}^2e\ {}^2c};\\
&\quad \ldots\\
\widehat{R}^{{}^{k-1}\alpha}{}_{{}^{k-1}\tau\ {}^{k-1}\beta\ {}^{k-1}\gamma} &= e_{\ {}^{k-1}\gamma}\widehat{L}^{{}^{k-1}\alpha}{}_{{}^{k-1}\tau\ {}^{k-1}\beta} - e_{{}^{k-1}\beta}\widehat{L}^{{}^{k-1}\alpha}{}_{{}^{k-1}\tau\ {}^{k-1}\gamma}\\
&\quad + \widehat{L}^{{}^{k-1}\mu}{}_{{}^{k-1}\tau\ {}^{k-1}\beta}\widehat{L}^{{}^{k-1}\alpha}{}_{{}^{k-1}\mu\ {}^{k-1}\gamma} - \widehat{L}^{{}^{k-1}\mu}{}_{{}^{k-1}\tau\ {}^{k-1}\gamma}\widehat{L}^{{}^{k-1}\alpha}{}_{{}^{k-1}\mu\ {}^{k-1}\beta}
\end{aligned}
$$

$$
\begin{aligned}
& - \widehat{C}^{k-1_\alpha}{}_{k-1_\tau\, k_a}\Omega^{k_a}{}_{k-1_\gamma\, k-1_\beta},\\
\widehat{R}^{k_a}{}_{k_b\, k-1_\beta\, k-1_\gamma} &= e_{k-1_\gamma}\widehat{L}^{k-k_a}{}_{k-1_b\, k-1_\beta} - e_{k-1_\beta}\widehat{L}^{k_a}{}_{k_b\, k-1_\gamma}\\
&\quad + \widehat{L}^{k_c}{}_{k_b\, k-1_\beta}\widehat{L}^{k_a}{}_{k_c\, k-1_\gamma} - \widehat{L}^{k_c}{}_{k_b\, k-1_\gamma}\widehat{L}^{k_a}{}_{k_c\, k-1_\beta}\\
&\quad - \widehat{C}^{k_a}{}_{k_b\, k_c}\Omega^{k_c}{}_{k-1_\gamma\, k-1_\beta},\\
\widehat{R}^{k-1_\alpha}{}_{k-1_\beta\, k-1_\gamma\, k_a} &= e_{k_a}\widehat{L}^{k-1_\alpha}{}_{k-1_\beta\, k-1_\gamma} - \widehat{D}_{k-1_\gamma}\widehat{C}^{k-1_\alpha}{}_{k-1_\beta\, k_a}\\
&\quad + \widehat{C}^{k-1_\alpha}{}_{k-1_\beta\, k_b}\widehat{T}^{k_b}{}_{k-1_\gamma\, k_a},\\
\widehat{R}^{k_c}{}_{k_b\, k-1_\gamma\, k_a} &= e_{k_a}\widehat{L}^{k_c}{}_{k_b\, k-1_\gamma} - \widehat{D}_{k-1_\gamma}\widehat{C}^{k_c}{}_{k_b\, k_a} + \widehat{C}^{k_c}_{k\ b\, k_d}\widehat{T}^{k_c}{}_{k-1_\gamma\, k_a},\\
\widehat{R}^{k-1_\alpha}{}_{k-1_\beta\, k_b\, k_c} &= e_{k_c}\widehat{C}^{\alpha}{}_{k-1_\beta\, k_b} - e_{k_b}\widehat{C}^{k-1_\alpha}{}_{k-1_\beta\, k_c}\\
&\quad + \widehat{C}^{k-1_\mu}{}_{k-1_\beta\, k_b}\widehat{C}^{k-1_\alpha}{}_{k-1_\mu\, k_c} - \widehat{C}^{k-1_\mu}{}_{k-1_\beta\, k_c}\widehat{C}^{k-1_\alpha}{}_{k-1_\mu\, k_b},\\
\widehat{R}^{k_a}{}_{k_b\, k_c\, k_d} &= e_{k_d}\widehat{C}^{k_a}{}_{k_b\, k_c} - e_{k_c}\widehat{C}^{k_a}{}_{k_b\, k_d}\\
&\quad + \widehat{C}^{k_e}{}_{k_b\, k_c}\widehat{C}^{k_a}{}_{k_e\, k_d} - \widehat{C}^{k_e}{}_{k_b\, k_d}\widehat{C}^{k_a}{}_{k_e\, k_c}.
\end{aligned}
$$

Proof It follows from "shell by shell computations" as in Refs. [1–4, 6–9, 30–32, 34–36]. □

Appendix B: Proof of Theorem 3.1

Such a proof can be obtained by straightforward computations as in Parts I and II of monograph [3], containing all developments from Refs. [30–32], see also summaries and some important details and discussions in Refs. [1, 2]. In this section, we generalize some formulas by considering "shell" labels for indices, when $k = 0, 1, 2, \dots$ using data (15) for a metric ${}^k\mathbf{g} = \{\mathbf{g}_{k_\beta\, k_\gamma}\}$ (14).

We can perform a N-adapted differential calculus on a N-anholonomic manifold if instead of partial derivatives $\partial_{k_\alpha} = \partial/\partial u^{k_\alpha}$ there are considered operators (10) parametrized in the form $\mathbf{e}_{k-1_\alpha} = \partial_{k-1_\alpha} - N^{k_a}_{k-1_\alpha}\partial_{k_a} = \partial_{k-1_\alpha} - w_{k-1_\alpha}\partial_{k_v} - n_{k-1_\alpha}\partial_{k_y}$, for $y^{4+2k} = {}^k v$ and $y^{5+2k} = {}^k y$. For instance, for data (15), the coefficients of N-connection curvature (13) are

$$
\begin{aligned}
\Omega^{4+2k}_{k-1_\alpha\, k-1_\beta} &= \partial_{k-1_\alpha} w_{k-1_\beta} - \partial_{k-1_\beta} w_{k-1_\alpha}\\
&\quad - w_{k-1_\alpha}\partial_{k_v} w_{k-1_\beta} + w_{k-1_\beta}\partial_{k_v} w_{k-1_\alpha};\\
\Omega^{5+2k}_{k-1_\alpha\, k-1_\beta} &= \partial_{k-1_\alpha} n_{k-1_\beta} - \partial_{k-1_\beta} n_{k-1_\alpha}\\
&\quad - w_{k-1_\alpha}\partial_{k_v} n_{k-1_\beta} + w_{k-1_\beta}\partial_{k_v} n_{k-1_\alpha}.
\end{aligned}
\tag{52}
$$

In a similar form we compute all coefficients of the canonical d-connection (17) and its Ricci and Einstein tensors.

B.1 Coefficients of the Canonical d-connection

For data (15), we get such nontrivial coefficients of $\widehat{\mathbf{\Gamma}}^{k_\gamma}{}_{k_\alpha k_\beta}$:

$$
\begin{aligned}
&\widehat{L}^2_{22} = \frac{\partial_2 g_2}{2g_2}, \qquad \widehat{L}^2_{23} = \frac{\partial_3 g_2}{2g_2}, \qquad \widehat{L}^2_{33} = -\frac{\partial_2 g_3}{2g_2}, \qquad \widehat{L}^3_{22} = -\frac{\partial_3 g_2}{2g_3},\\
&\widehat{L}^3_{23} = \frac{\partial_2 g_3}{2g_3}, \qquad \widehat{L}^3_{33} = \frac{\partial_3 g_3}{2g_3}, \qquad \widehat{L}^4_{4i} = \frac{1}{2h_4}(\partial_i h_4 - w_i \partial_v h_4); \qquad \widehat{L}^5_{4j} = \frac{1}{2}\partial_v n_j,\\
&\widehat{L}^5_{5j} = \frac{1}{2h_5}(\partial_j h_5 - w_j \partial_v h_5); \qquad \widehat{C}^4_{44} = \frac{\partial_v h_4}{2h_4}, \qquad \widehat{C}^4_{55} = -\frac{\partial_v h_5}{2h_4}, \qquad \widehat{C}^5_{45} = \frac{\partial_v h_5}{2h_5};\\
&\dots\\
&\widehat{L}^{4+2k}_{4+2k\ k-1_\alpha} = \frac{1}{2h_{4+2k}}(\partial_{k-1_\alpha} h_{4+2k} - w_{k-1_\alpha}\partial_{k_v} h_{4+2k}),\\
&\widehat{L}^{5+2k}_{4+2k\ k-1_\alpha} = \frac{1}{2}\partial_{k_v} n_{k-1_\alpha}, \qquad \widehat{L}^{5+2k}_{5+2k\ k-1_\alpha} = \frac{1}{2h_{5+2k}}(\partial_{k-1_\alpha} h_{5+2k} - w_{k-1_\alpha}\partial_{k_v} h_{5+2k});\\
&\widehat{C}^{4+2k}_{4+2k\ 4+2k} = \frac{\partial_{k_v} h_{4+2k}}{2h_{4+2k}}, \qquad \widehat{C}^{4+2k}_{5+2k\ 5+2k} = -\frac{\partial_{k_v} h_{5+2k}}{2h_{4+2k}},\\
&\widehat{C}^{5+2k}_{4+2k\ 5+2k} = \frac{\partial_{k_v} h_{5+2k}}{2h_{5+2k}}.
\end{aligned}
\tag{53}
$$

We note that

$$
\widehat{C}^i_{jc} = \frac{1}{2} g^{ik} \frac{\partial g_{jk}}{\partial y^c} = 0, \quad \dots, \qquad \widehat{C}^{k-1_\alpha}_{k-1_\beta\ k_c} = \frac{1}{2} g^{k-1_\alpha\ k-1_\tau} \frac{\partial g_{k-1_\beta\ k-1_\tau}}{\partial y^{k_c}} = 0, \tag{54}
$$

which is an important condition for generating exact solutions of the Einstein equations for the Levi-Civita connection, see formulas (30).

B.2 Calculation of Torsion Coefficients

The nontrivial coefficients of torsions (20) for data (15) are given by formulas (52) and, respectively, (53) resulting in

$$
\begin{aligned}
\widehat{T}^{4+2k}_{k-1_\alpha\ k-1_\beta} &= \partial_{k-1_\beta} w_{k-1_\alpha} - \partial_{k-1_\alpha} w_{k-1_\beta}\\
&\quad - w_{k-1_\beta}\partial_{k_v} w_{k-1_\alpha} + w_{k-1_\alpha}\partial_{k_v} w_{k-1_\beta};\\
\widehat{T}^{5+2k}_{k-1_\alpha\ k-1_\beta} &= \partial_{k-1_\beta} n_{k-1_\alpha} - \partial_{k-1_\alpha} n_{k-1_\beta}\\
&\quad w_{k-1_\beta}\partial_{k_v} n_{k-1_\alpha} - w_{k-1_\alpha}\partial_{k_v} n_{k-1_\beta},\\
\widehat{T}^{4+2k}_{4+2k\ k-1_\alpha} &= \partial_{k_v} w_{k-1_\alpha} - \frac{1}{2h_{4+2k}}(\partial_{k-1_\alpha} h_{4+2k} - w_{k-1_\alpha}\partial_{k_v} h_{4+2k}),\\
\widehat{T}^{4+2k}_{5+2k\ k-1_\alpha} &= \frac{h_{5+2k}}{2h_{4+2k}}\partial_{k_v} n_{k-1_\alpha}, \qquad \widehat{T}^{5+2k}_{4+2k\ k-1_\alpha} = \frac{1}{2}\partial_{k_v} n_{k-1_\alpha},\\
\widehat{T}^{5+2k}_{5+2k\ k-1_\alpha} &= -\frac{1}{2h_{5+2k}}(\partial_{k-1_\alpha} h_{5+2k} - w_{k-1_\alpha}\partial_{k_v} h_{5+2k}).
\end{aligned}
\tag{55}
$$

B.3 Calculation of the Ricci Tensor

For instance, let us compute the values $\widehat{R}_{ij} = \widehat{R}^k{}_{ijk}$ from (26),

$$\widehat{R}^i{}_{hjk} = \mathbf{e}_k \widehat{L}^i_{.hj} - \mathbf{e}_j \widehat{L}^i_{.hk} + \widehat{L}^m_{.hj}\widehat{L}^i_{mk} - \widehat{L}^m_{.hk}\widehat{L}^i_{mj} - \widehat{C}^i_{.ha}\Omega^a_{.jk},$$

using (51) and $\widehat{C}^i_{.ha} = 0$ (54). We have $\mathbf{e}_k \widehat{L}^i_{.hj} = \partial_k \widehat{L}^i_{.hj} + N^a_k \partial_a \widehat{L}^i_{.hj} = \partial_k \widehat{L}^i_{.hj} + w_k (\widehat{L}^i_{.hj})^* = \partial_k \widehat{L}^i_{.hj}$ because $\widehat{L}^i_{.hj}$ do not depend on variable $y^4 = v$. We use, in brief, denotations of type $\partial_2 g = g^\bullet$, $\partial_3 g = g^{'}$, $\partial_4 g = g^*$.

Deriving (53), we obtain

$$\partial_2 \widehat{L}^2{}_{22} = \frac{g_2^{\bullet\bullet}}{2g_2} - \frac{(g_2^\bullet)^2}{2(g_2)^2}, \qquad \partial_2 \widehat{L}^2{}_{23} = \frac{g_2^{\bullet '}}{2g_2} - \frac{g_2^\bullet g_2^{'}}{2(g_2)^2},$$

$$\partial_2 \widehat{L}^2{}_{33} = -\frac{g_3^{\bullet\bullet}}{2g_2} + \frac{g_2^\bullet g_3^\bullet}{2(g_2)^2}, \qquad \partial_2 \widehat{L}^3{}_{22} = -\frac{g_2^{\bullet '}}{2g_3} + \frac{g_2^\bullet g_3^{'}}{2(g_3)^2},$$

$$\partial_2 \widehat{L}^3{}_{23} = \frac{g_3^{\bullet\bullet}}{2g_3} - \frac{(g_3^\bullet)^2}{2(g_3)^2}, \qquad \partial_2 \widehat{L}^3{}_{33} = \frac{g_3^{\bullet '}}{2g_3} - \frac{g_3^\bullet g_3^{'}}{2(g_3)^2},$$

$$\partial_3 \widehat{L}^2{}_{22} = \frac{g_2^{\bullet '}}{2g_2} - \frac{g_2^\bullet g_2^{'}}{2(g_2)^2}, \qquad \partial_3 \widehat{L}^2{}_{23} = \frac{g_2^{ll}}{2g_2} - \frac{(g_2^l)^2}{2(g_2)^2},$$

$$\partial_3 \widehat{L}^2{}_{33} = -\frac{g_3^{\bullet '}}{2g_2} + \frac{g_3^\bullet g_2^{'}}{2(g_2)^2}, \qquad \partial_3 \widehat{L}^3{}_{22} = -\frac{g_2^{''}}{2g_3} + \frac{g_2^\bullet g_2^{'}}{2(g_3)^2},$$

$$\partial_3 \widehat{L}^3{}_{23} = \frac{g_3^{\bullet '}}{2g_3} - \frac{g_3^\bullet g_3^{'}}{2(g_3)^2}, \qquad \partial_3 \widehat{L}^3{}_{33} = \frac{g_3^{ll}}{2g_3} - \frac{(g_3^l)^2}{2(g_3)^2}.$$

For these values, there are only 2 nontrivial components,

$$\widehat{R}^2{}_{323} = \frac{g_3^{\bullet\bullet}}{2g_2} - \frac{g_2^\bullet g_3^\bullet}{4(g_2)^2} - \frac{(g_3^\bullet)^2}{4g_2 g_3} + \frac{g_2^{ll}}{2g_2} - \frac{g_2^l g_3^l}{4g_2 g_3} - \frac{(g_2^l)^2}{4(g_2)^2},$$

$$\widehat{R}^3{}_{223} = -\frac{g_3^{\bullet\bullet}}{2g_3} + \frac{g_2^\bullet g_3^\bullet}{4g_2 g_3} + \frac{(g_3^\bullet)^2}{4(g_3)^2} - \frac{g_2^{ll}}{2g_3} + \frac{g_2^l g_3^l}{4(g_3)^2} + \frac{(g_2^l)^2}{4g_2 g_3}$$

with $\widehat{R}_{22} = -\widehat{R}^3{}_{223}$ and $\widehat{R}_{33} = \widehat{R}^2{}_{323}$, or

$$\widehat{R}^2_2 = \widehat{R}^3_3 = -\frac{1}{2g_2 g_3}\left[g_3^{\bullet\bullet} - \frac{g_2^\bullet g_3^\bullet}{2g_2} - \frac{(g_3^\bullet)^2}{2g_3} + g_2^{''} - \frac{g_2^l g_3^l}{2g_3} - \frac{(g_2^l)^2}{2g_2} \right]$$

as in (32).

Now, we consider

$$\widehat{R}^c{}_{bka} = \frac{\partial \widehat{L}^c_{.bk}}{\partial y^a} - \left(\frac{\partial \widehat{C}^c_{.ba}}{\partial x^k} + \widehat{L}^c_{.dk}\widehat{C}^d_{.ba} - \widehat{L}^d_{.bk}\widehat{C}^c_{.da} - \widehat{L}^d_{.ak}\widehat{C}^c_{.bd} \right) + \widehat{C}^c_{.bd}\widehat{T}^d_{.ka}$$

$$= \frac{\partial \widehat{L}^c_{.bk}}{\partial y^a} - \widehat{C}^c_{.ba|k} + \widehat{C}^c_{.bd}\widehat{T}^d_{.ka}$$

from (51). Contracting indices, we get $\widehat{R}_{bk}=\widehat{R}^{a}{}_{bka}=\frac{\partial L^{a}_{.bk}}{\partial y^{a}}-\widehat{C}^{a}_{.ba|k}+\widehat{C}^{a}_{.bd}\widehat{T}^{d}_{.ka}$. Let us denote $\widehat{C}_{b}=\widehat{C}^{c}_{.ba}$ and write $\widehat{C}_{.b|k}=\mathbf{e}_{k}\widehat{C}_{b}-\widehat{L}^{d}{}_{bk}\widehat{C}_{d}=\partial_{k}\widehat{C}_{b}-N^{e}_{k}\partial_{e}\widehat{C}_{b}-\widehat{L}^{d}{}_{bk}\widehat{C}_{d}=\partial_{k}\widehat{C}_{b}-w_{k}\widehat{C}^{*}_{b}-\widehat{L}^{d}{}_{bk}\widehat{C}_{d}$. We express $\widehat{R}_{bk}={}_{[1]}R_{bk}+{}_{[2]}R_{bk}+{}_{[3]}R_{bk}$, where

$$
\begin{aligned}
{}_{[1]}R_{bk}&=(\widehat{L}^{4}_{bk})^{*}, \qquad {}_{[2]}R_{bk}=-\partial_{k}\widehat{C}_{b}+w_{k}\widehat{C}^{*}_{b}+\widehat{L}^{d}{}_{bk}\widehat{C}_{d},\\
{}_{[3]}R_{bk}&=\widehat{C}^{a}_{.bd}\widehat{T}^{d}_{.ka}=\widehat{C}.b4^{4}\widehat{T}^{4}_{.k4}+\widehat{C}^{4}_{.b5}\widehat{T}^{5}_{.k4}+\widehat{C}^{5}_{.b4}\widehat{T}^{4}_{.k5}+\widehat{C}^{5}_{.b5}\widehat{T}^{5}_{.k5}
\end{aligned}
$$

for $\widehat{C}_{4}=\widehat{C}^{4}_{44}+\widehat{C}^{5}_{45}=\frac{h^{*}_{4}}{2h_{4}}+\frac{h^{*}_{5}}{2h_{5}}, \widehat{C}_{5}=\widehat{C}^{4}_{54}+\widehat{C}^{5}_{55}=0.$

We compute $\widehat{R}_{4k}={}_{[1]}R_{4k}+{}_{[2]}R_{4k}+{}_{[3]}R_{4k}$ with

$$
\begin{aligned}
{}_{[1]}R_{4k}&=(\widehat{L}^{4}_{4k})^{*}, \qquad {}_{[2]}R_{4k}=-\partial_{k}\widehat{C}_{4}+w_{k}\widehat{C}^{*}_{4}+\widehat{L}^{4}{}_{4k}\widehat{C}_{4},\\
{}_{[3]}R_{4k}&=\widehat{C}^{4}_{.44}\widehat{T}^{4}_{.k4}+\widehat{C}^{4}_{.45}\widehat{T}^{5}_{.k4}+\widehat{C}^{5}_{.44}\widehat{T}^{4}_{.k5}+\widehat{C}^{5}_{.45}\widehat{T}^{5}_{.k5}.
\end{aligned}
$$

Summarizing, we get

$$
2h_{5}\widehat{R}_{4k}=w_{k}\left[h^{**}_{5}-\frac{(h^{*}_{5})^{2}}{2h_{5}}-\frac{h^{*}_{4}h^{*}_{5}}{2h_{4}}\right]+\frac{h^{*}_{5}}{2}\left(\frac{\partial_{k}h_{4}}{h_{4}}+\frac{\partial_{k}h_{5}}{h_{5}}\right)-\partial_{k}h^{*}_{5}
$$

which is equivalent to (34).

In a similar way, we compute $\widehat{R}_{5k}={}_{[1]}R_{5k}+{}_{[2]}R_{5k}+{}_{[3]}R_{5k}$, where

$$
\begin{aligned}
{}_{[1]}R_{5k}&=(\widehat{L}^{4}_{5k})^{*}, \qquad {}_{[2]}R_{5k}=-\partial_{k}\widehat{C}_{5}+w_{k}\widehat{C}^{*}_{5}+\widehat{L}^{4}{}_{5k}\widehat{C}_{4},\\
{}_{[3]}R_{5k}&=\widehat{C}^{4}_{.54}\widehat{T}^{4}_{.k4}+\widehat{C}^{4}_{.55}\widehat{T}^{5}_{.k4}+\widehat{C}^{5}_{.54}\widehat{T}^{4}_{.k5}+\widehat{C}^{5}_{.55}\widehat{T}^{5}_{.k5}.
\end{aligned}
$$

We have $\widehat{R}_{5k}=(\widehat{L}^{4}_{5k})^{*}+\widehat{L}^{4}{}_{5k}\widehat{C}_{4}+\widehat{C}^{4}_{.55}\widehat{T}^{5}_{.k4}+\widehat{C}^{5}_{.54}\widehat{T}^{4}_{.k5}=(-\frac{h_{5}}{h_{4}}n^{*}_{k})^{*}-\frac{h_{5}}{h_{4}}n^{*}_{k}(\frac{h^{*}_{4}}{2h_{4}}+\frac{h^{*}_{5}}{2h_{5}})+\frac{h^{*}_{5}}{2h_{5}}\frac{h_{5}}{2h_{4}}n^{*}_{k}-\frac{h^{*}_{5}}{2h_{4}}\frac{1}{2}n^{*}_{k}$, which can be written

$$
2h_{4}\widehat{R}_{5k}=h_{5}n^{**}_{k}+\left(\frac{h_{5}}{h_{4}}h^{*}_{4}-\frac{3}{2}h^{*}_{5}\right)n^{*}_{k},
$$

i.e. we prove (35).

For

$$
\widehat{R}^{i}{}_{jka}=\frac{\partial\widehat{L}^{i}_{.jk}}{\partial y^{k}}-\left(\frac{\partial\widehat{C}^{i}_{.ja}}{\partial x^{k}}+\widehat{L}^{i}_{.lk}\widehat{C}^{l}_{.ja}-\widehat{L}^{l}_{.jk}\widehat{C}^{i}_{.la}-\widehat{L}^{c}_{.ak}\widehat{C}^{i}_{.jc}\right)+\widehat{C}^{i}_{.jb}\widehat{T}^{b}_{.ka}
$$

from (51), we obtain zeros because $\widehat{C}^{i}_{.jb}=0$ and $\widehat{L}^{i}_{.jk}$ do not depend on y^{k}. So, $\widehat{R}_{ja}=\widehat{R}^{i}{}_{jia}=0$.

Taking $\widehat{R}^{a}{}_{bcd}=\frac{\partial\widehat{C}^{a}_{.bc}}{\partial y^{d}}-\frac{\partial\widehat{C}^{a}_{.bd}}{\partial y^{c}}+\widehat{C}^{e}_{.bc}\widehat{C}^{a}_{.ed}-\widehat{C}^{e}_{.bd}\widehat{C}^{a}_{.ec}$ from (51) and contracting the indices in order to obtain the Ricci coefficients, $\widehat{R}_{bc}=\frac{\partial\widehat{C}^{d}_{.bc}}{\partial y^{d}}-\frac{\partial\widehat{C}^{d}_{.bd}}{\partial y^{c}}+\widehat{C}^{e}_{.bc}\widehat{C}^{d}_{.ed}-\widehat{C}^{e}_{.bd}\widehat{C}^{d}_{.ec}$, we compute

$$
\widehat{R}_{bc}=(\widehat{C}^{4}_{.bc})^{*}-\partial_{c}\widehat{C}_{b}+\widehat{C}^{4}_{.bc}\widehat{C}_{4}-\widehat{C}^{4}_{.b4}\widehat{C}^{4}_{.4c}-\widehat{C}^{4}_{.b5}\widehat{C}^{5}_{.4c}-\widehat{C}^{5}_{.b4}\widehat{C}^{4}_{.5c}-\widehat{C}^{5}_{.b5}\widehat{C}^{5}_{.5c}.
$$

There are nontrivial values, $\widehat{R}_{44}=(\widehat{C}^{4}_{.44})^{*}-\widehat{C}^{*}_{4}+\widehat{C}^{4}_{44}(\widehat{C}_{4}-\widehat{C}^{4}_{44})-(\widehat{C}^{5}_{.45})^{2}$ and $\widehat{R}_{55}=(\widehat{C}^{4}_{.55})^{*}-\widehat{C}^{4}_{.55}(-\widehat{C}_{4}+2\widehat{C}^{5}_{.45})$ resulting in

$$
\widehat{R}^{4}_{4}=\widehat{R}^{5}_{5}=\frac{1}{2h_{4}h_{5}}\left[-h^{**}_{5}+\frac{(h^{*}_{5})^{2}}{2h_{5}}+\frac{h^{*}_{4}h^{*}_{5}}{2h_{4}}\right]
$$

which is just (33).

Computations for higher shells, with $k = 1, 2, \ldots$ are similar.

Theorem 3.1 is proven.

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